Generalization of the Differentiation Process.

By Robert E. Moritz.

1.—Notation and Definitions.

For the purpose of this discussion, we shall use the following notation:

The symbol) represents any process or operation whatsoever between any two concepts a and b, thus:

$$a\ )\ b = c,$$

where c is the result of the operation, a is the operand, b the operator of the process. If we look upon a as the operator and upon b as the operand, we write

$$b\ (\ a = c,$$

where the symbol (represents, of course, a new operation.

For every combinatory process), there exist, in general, two dissociating processes known as inverses. We denote these by the symbols $\smile$ and $\frown$, the former denoting the dissociation of c and b, the latter the dissociation of c and a, thus:

$$c \smile b = a, \qquad c \frown a = b.$$

and we refer to these processes as the first and second inverses of) respectively. Hence, they are likewise the second and first inverses of (respectively.

We may look upon c and b as the original concepts, and upon a as the result obtained from their combination, that is, we may look upon either inverse as the direct process and upon the former direct process as one of the inverses of the inverse.

If, among the process and its inverses, there is one which is single valued, that one is selected as the direct process. In this paper we shall assume throughout the singlevaluedness of the direct processes.

In special cases) and (may denote the same operation, and in such cases we shall represent the operation by |, a symbol devoid of aspect. It immediately follows that in this case $\smile$ and $\frown$ will also be the same operation which we will denote by —.

When relations are considered which subsist not only for the direct processes) and (, but for their inverses $\smile$ and $\frown$ as well, the symbol $\underset{\frown}{\overset{\smile}{)(}}$ might be employed, this symbol being merely a combination of the others; in cases where the two inverses are identical, the symbol is contracted into $\top$. This latter form corresponds to the well-known symbols $\mp$, $\times$ of the ordinary algebra. When we deal with a second operation, we represent it by)) and its inverses by $\underset{\smile}{\smile}$ and $\overset{\frown}{\frown}$ respectively. To the symbol | corresponds || and to $\top$ corresponds $\overline{\overline{||}}$.*

We shall have occasion to deal with associated processes, processes which are related to each other by definite laws. The addition, multiplication, and involution processes of the ordinary algebra are such processes. We shall distinguish associated processes from each other by using as symbols for them) with various indices. Suppose that we have selected a reference process, say

$$a\,)_0\,b = c,$$

then we may write

$$\begin{aligned}
&a\,)_0\,a\,)_0 \;\ldots\ldots \text{ to } \omega_0(b) \text{ terms} = a\,)_1\,b,\\
&a\,)_1\,a\,)_1 \;\ldots\ldots \text{ `` } \omega_1(b) \quad \text{``} \quad = a\,)_2\,b,\\
&\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&a\,)_n\,a\,)_n \;\ldots\ldots \text{ `` } \omega_n(b) \quad \text{``} \quad = a\,)_{n+1}\,b,
\end{aligned}$$

where $\omega_0(b)$, $\omega_1(b)$, ..., $\omega_n(b)$ represent arbitrarily selected functions of b. We shall refer to such processes as processes of the zero, first, second, ... n^{th}, $(n+1)^{\text{th}}$ orders respectively, the reference process being of the zero order.

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* I am aware that I am guilty of adding another set of symbols to the already too long list of symbols with a similar meaning used by others. Grassmann uses $\frown \smile$ and also $\overset{\frown}{\frown} \underset{\smile}{\smile}$ to denote direct and inverse combinations respectively, De Morgan uses $\lambda\ \gamma$, Hankel $\lambda\ \theta$, Stolz $\bigcirc \frown$ and also $\odot \,\hat{}$, Houel $\frown \smile$ and $\Uparrow \Downarrow$. Davis suggests the use of any letter which has aspect as U, Y, or V; $a \curlyvee b$ or $b \curlywedge a$ could be used to denote any combination between a and b, $a \prec b$ and $a \succ b$ its two inverses, ψ ψ, and φ φ have also been suggested, and there are no doubt many others which I have not seen. Where there are so many symbols to choose from, it is easier to devise a new symbol than to discriminate justly among the old.

When $\omega_0(b), \omega_1(b), \ldots\ldots \omega_n(b)$ become all equal to b, $)_0$ may be taken equal to $+$, and $)_1$, $)_2$ become the $\times$, $(\ )^{(\ )}$ of the ordinary algebra.

Any process may be selected as the reference process and negative indices may be used as well as positive. Thus we may have processes defined as follows:

$$\begin{array}{llll} a\,)_{-n} \quad a\,)_{-n} & \ldots\ldots \text{ to } \omega_{-n}(b) & \text{terms} & = a\,)_{-n+1}\, b, \\ a\,)_{-n+1} a\,)_{-n+1} & \ldots\ldots \text{ ``} \ \omega_{-n+1}(b) & \text{``} & = a\,)_{-n+2}\, b, \\ \vdots & \quad\vdots & & \quad\vdots \\ a\,)_{-1} \quad a\,)_{-1} & \ldots\ldots \text{ ``} \ \omega_{-1}(b) & \text{``} & = a\,)_0\, b, \end{array}$$

which shows how the process $a\,)_0\, b$ and hence any higher process may be defined in terms of any negative process. The first and second inverses of $)_n$ are, of course, $\smile_n$ and $\frown_n$ respectively.

The laws of combination, to which different operations are subject, cannot be determined *a priori*. They depend upon the nature of the concepts involved, the relations which subsist among the combinations employed, and upon the meanings attached to the results of the combinations. In a study of pure form, we may posit any laws of combination and examine the consequences following their assumption. In that case the concepts, among which the combinations take place, are in part defined by these laws, and hence may or may not have an existence in the world of experience. On the other hand, we may limit our discussion to such laws of combination and their consequences as are suggested by experience. There are five such laws which may be put under three general heads and are defined as follows:

1.—*The Commutative Law*, $a\,)\, b = b\,)\, a$. This law is characteristic of any process denoted by $|$ or $\|$.

2.—*The Associative Law*, $a\,)_n b\,)_n c = a\,)_n (b\,)_m c)$, or when it is expressed in the most general form possible

$$f(a, b, c) = \phi\,[a, \psi(b, c)].$$

3.—*The Distributive Law*, the common form of which comes under the form $a\,)_n (b\,)_p\, c) = a\,)_n\, b\,)_q (a\,)_n\, c)$, and this is but a special case of the still more general form

$$F\,[a, f(b, c)] = \Phi\,[\phi\,(a, b), \psi\,(a, c)].$$

Among the symbols which enter into combination with each other, there is frequently one, M_n, which satisfies the equation $a\,)_n M_n = a$, no matter what a is. Such a symbol is called the *modulus* of the process under consideration. The

second inverse gives immediately $a \frown_n a = M_n$. Also $a \smile_n M_n = a$, that is, M_n is likewise a modulus of the first inverse.

2.—PRELIMINARY THEOREMS.

Grassmann,* Hankel,† and Stolz‡ have considered at length single associative processes both with and without commutation. Interesting results have been reached by them, as for instance that the formulæ

$$a \smile (b \,)\, c) = a \smile b \smile c \quad \text{and} \quad a \smile (b \smile c) = a \smile b \,)\, c$$

involve the commutative law, and that a combinatory process which is commutative for two terms and associative for three, is commutative and associative for any number of terms. Equally interesting is the study of the theory of associated processes. We shall prove three theorems.

THEOREM I.—*If an associated process of the n^{th} order admits a modulus M_n, then $M_n \frown_{n+1} M_n$ is indeterminate.*

We have, by definition and hypothesis,

$$a \,)_n\, M_n = a,$$

and, therefore,

$$a \,)_n\, M_n \,)_n\, M_n \,)_n \ldots \text{ to } f(b) \text{ terms} = a.$$

a may be taken equal to M_n, hence,

$$M_n \,)_n\, M_n \,)_n \ldots \text{ to } f(b) \text{ terms} = M_n,$$

that is,

$$M_n \,)_{n+1}\, b = M_n.$$

The second inverse of this last equation yields

$$M_n \frown_{n+1} M_n = b, \text{ no matter what } b \text{ is.}$$

Familiar illustrations of this theorem from the ordinary algebra are the forms $\frac{0}{0}$ and $\log_1 1$.

* Grassmann, "Ausdehnungslehre," 1844.

† Hankel, "Vorlesungen über Complexe Zahlen."

‡ Stolz, "Algemeine Arithmetik," Bd. 1.

THEOREM II.—*If two associated processes are subject to an associative law, then*

$$p \frown_n q = p \frown_n a \frown_m (q \frown_n a)\,;$$

if, moreover, one of the processes admits a modulus, then

$$p \frown_n q = M_n \frown_m (q \frown_n p).$$

The proof is simple. Let

$$a\,)_n\, b\,)_n\, c = p,$$

and

$$a\,)_n\, b = q,$$

then will

$$q\,)_n\, c = p.$$

By hypothesis

$$a\,)_n\, b\,)_n\, c = a\,)_n\, (b\,)_m\, c).$$

Let

$$b\,)_m\, c = r,$$

then

$$a\,)_n\, r = p.$$

The second inverses of these processes are

$$\begin{aligned} q \frown_n a &= b, \\ p \frown_n q &= c, \\ r \frown_m b &= c, \\ p \frown_n a &= r. \end{aligned}$$

Now substitute in the third of the last four equations, the values of r, b and c, from the other three, and we get

$$p \frown_n a \frown_m (q \frown_n a) = p \frown_n q.$$

If, now, the n^{th} process admits a modulus, we have, by putting p for a in the last equation,

$$p \frown_n p \frown_m (q \frown_n p) = p \frown_n q,$$

or

$$M_n \frown_m (q \frown_n p) = p \frown_n q.$$

If, on the other hand, the m^{th} process has a modulus, we have, by putting p for q

$$p \frown_n a \frown_m (p \frown_n a) = p \frown_n p,$$

or

$$M_m = p \frown_n p.$$

This last equation holds for any p, hence the n^{th} process has a modulus, and the theorem follows as before.

In the ordinary algebra, the m in our statement of the associative principle

is either n or $n-1$, and hence our theorem gives the forms,

for $m=n$,

$$p \frown_n q = p \frown_n a \frown_n (q \frown_n a),$$
$$p \frown_n q = M_n \frown_n (q \frown_n p),$$

for $m=n-1$,

$$p \frown_n q = p \frown_n a \frown_{n-1} (q \frown_n a),$$
$$p \frown_n q = M_n \frown_{n-1} (q \frown_n p).$$

For $n=0$, 1 and 2 successively, we get the familiar equations

$$p-q=p-a-(q-a), \qquad \log_q p = \frac{\log_a p}{\log_a q};$$
$$p \div q = p \div a \div (q \div a);$$

and

$$p-q=-(q-p), \qquad \log_q p = \frac{1}{\log_p q}.$$
$$p \div q = 1 \div (q \div p).$$

THEOREM III.—*If three associated processes, of the* $(n-1)^{\text{th}}$, n^{th} *and* m^{th} *orders respectively, of which the first and third admit each a modulus,* M_{n-1} *and* M_m, *are subject to a distributive law, then* $M_{n-1} \frown_n a = M_m$, *a being different from* M_{n-1}.

By hypothesis,

$$a\,)_n\,(b\,)_m\,c) = a\,)_n\,b\,)_{n-1}\,(a\,)_n\,c),$$

and, therefore,

$$a\,)_n\,c = a\,)_n\,(b\,)_m\,c) \frown_{n-1} (a\,)_n\,b).$$

Now put $b\,)_m\,c = b'$ and $c = b' \frown_m b$, and the last equation becomes

$$a\,)_n\,(b' \frown_m b) = a\,)_n\,b' \frown_{n-1} (a\,)_n\,b).$$

Let $b'=b$, $b' \frown_m b$ becomes M_m and the right member of the last equation becomes M_{n-1}, so that

$$a\,)_n\,M_m = M_{n-1}, \text{ unless } a = M_{n-1},$$

or, finally,

$$M_{n-1} \frown_n a = M_m.$$

If $a = M_{n-1}$, the left member of the last equation becomes $M_{n-1} \frown_n M_{n-1}$, which by theorem I, is indeterminate.

In the ordinary algebra, the distributive law holds only for $m=n-1$ and $m=n-2$. We obtain then the two forms

$$M_{n-1} \frown_n a = M_{n-1}, \quad a \neq M_{n-1},$$

and

$$M_{n-1} \frown_n a = M_{n-2}, \quad a \neq M_{n-1},$$

which, for $n=1$ and $n=2$ successively, reduce to

$$\frac{0}{a} = 0, \quad a \neq 0; \quad \text{and} \quad \log_1^a = 0, \quad a \neq 1.$$

3.—LIMITING PROCESSES ALLIED TO DIFFERENTIATION.

Using the general notation defined in art. 1 instead of that ordinarily employed in writing the differential coefficient of a function, the definition equation of the differential coefficient may be written as follows,

$$\frac{dy}{dx} = \lim_{h=M_0} \{F(x\,)_0\, h) \frown_0 F(x) \frown_1 h\},$$

where $y = F(x)$.

This form of the differential coefficient suggests at once other processes which lead to the symbols

$$\frac{d_1 y}{d_1 x},\ \frac{d_2 y}{d_2 x},\ \ldots\ldots,\ \frac{d_n y}{d_n x},$$

as defined by the following equations,

$$\begin{aligned}
\frac{d_1 y}{d_1 x} &= \lim_{h=M_1} \{F(x\,)_1\, h) \frown_1 F(x) \frown_2 h\},\\
\frac{d_2 y}{d_2 x} &= \lim_{h=M_2} \{F(x\,)_2\, h) \frown_2 F(x) \frown_3 h\},\\
&\vdots\\
\frac{d_n y}{d_n x} &= \lim_{h=M_n} \{F(x\,)_n\, h) \frown_n F(x) \frown_{n+1} h\}.
\end{aligned}$$

The ordinary differential coefficient is the $\frac{d_0 y}{d_0 x}$ belonging to this chain of expressions.

Consider the process defined by the last equation. We pass to the limit indicated and have

$$\begin{aligned}
\frac{d_n y}{d_n x} &= \lim_{h=M_n} \{F(x\,)_n\, h) \frown_n F(x) \frown_{n+1} h\}\\
&= F(x\,)_n\, M_n) \frown_n F(x) \frown_{n+1} M_n\\
&= F(x) \frown_n F(x) \frown_{n+1} M_n = M_n \frown_{n+1} M_n.
\end{aligned}$$

The right member of the last equation is indeterminate by theorem I of the preceding section, and gives rise to an algebraic limiting process, provided that a process of the n^{th} order exists, and that this process admits a modulus. The evaluation of $\frac{d_n y}{d_n x}$ may be considered, and if it can be effected, will give rise to a set of forms of which the ordinary differential coefficients constitute a particular

set. In some cases the evaluation can be readily effected. If, for instance, an associative and distributive law obtains, as in the case of the involution process of the ordinary algebra, then

$$\frac{d_n y}{d_n x} = M_n \frown_{n+1} M_n = M_n \frown_{n+1} a \frown_n (M_n \frown_{n+1} a)$$

by aid of theorem II of the last section, and this in turn, by applying theorem III, may be transformed into

$$\frac{d_n y}{d_n x} = M_{n-1} \frown_n M_{n-1},$$

that is, $M_n \frown_{n+1} M_n$ can be evaluated, provided $M_{n-1} \frown_n M_{n-1}$ can be. In this case, and others that will be treated later, $\frac{d_n y}{d_n x}$ can be expressed as a function of $\frac{d_{n-1} y}{d_{n-1} x}$.

4.—QUOTIENTIAL COEFFICIENTS.

Let us now consider more closely the process defined by

$$\frac{d_1 y}{d_1 x} = \lim_{h = M_1} \{F(x)_1 h) \frown_1 F(x) \frown_2 h\}.$$

When the operations involved in this equation are those of the ordinary algebra, we shall write $\frac{qy}{qx}$ for $\frac{d_1 y}{d_1 x}$, so that

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \log_h \frac{F(xh)}{F(x)} \right\},$$

from which follows

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \frac{\log \frac{F(xh)}{F(x)}}{\log h} \right\}.$$

This is of the form $\frac{0}{0}$ and may hence be evaluated by well known rules. We have

$$\frac{qy}{qx} = \lim_{h=1} \left\{ \frac{h\, F'_h(xh)}{F(xh)} \right\}, \text{ where } F'_h(xh) = \frac{dF(xh)}{dh},$$

but

$$\lim_{h=1} \left\{ F'_h(xh) \right\} = x\frac{dFx}{dx} = xF'(x),$$

so that
$$\frac{qy}{qx}=\frac{xF'(x)}{F(x)}=\frac{x}{y}\cdot\frac{dy}{dx}=\frac{d\log y}{d\log x}.$$

Or we proceed as follows
$$\frac{qy}{qx}=\lim_{h=1}\left\{\log_h\frac{f(xh)}{f(x)}\right\}=\lim_{h=1}\left\{\frac{\log f(xh)-\log f(x)}{\log h}\right\}$$
$$=\lim_{h'=0}\left\{\frac{F(z+h')-F(z)}{h'}\right\},$$

where $h'=\log h$, $z=\log x$, $F(z)=\log f(x)$, hence,
$$\frac{qy}{qx}=\frac{dF(z)}{dz}=\frac{d\log f(x)}{d\log x}=\frac{d\log y}{d\log x}=\frac{x}{y}\cdot\frac{dy}{dx}.$$

We call the form thus derived the *quotiential coefficient* of the function $y=f(x)$, and refer to the process of deriving it as *quotientiation*. The following table gives a partial list of quotiential coefficients:

(1). $\frac{q(a)}{qx}=0$, a being any constant.

(2). $\frac{q(x)}{qx}=1$. (3). $\frac{q(ax^n)}{qx}=n$.

(4). $$\frac{q(ax^n\pm bx^m)}{qx}=\frac{nax^n\pm mbx^m}{ax^n\pm bx^m}.$$

(5). $\frac{q(\sin x)}{qx}=x\cot x$. (6). $\frac{q(a^{\phi(x)})}{qx}=x\log a\,.\,\phi'(x)$.

(7). $\frac{q(e^x)}{qx}=x$. (8). $\frac{q(\log x)}{qx}=\frac{1}{\log x}$.

(9). $$\frac{q(u+v+w+\dots)}{qx}=\frac{u\frac{qu}{qx}+v\frac{qv}{qx}+w\frac{qw}{qx}+\dots}{u+v+w+\dots},$$

$u, v, w, \dots$ being each a function of x.

(10). $$\frac{q(u\,.\,v\,.\,w\dots)}{qx}=\frac{qu}{qx}+\frac{qv}{qx}+\frac{qw}{qx}+\dots$$

(11). $\frac{q\left(\frac{u}{v}\right)}{qx}=\frac{qu}{qx}-\frac{qv}{qx}$. (12). $\frac{q\left(\frac{1}{u}\right)}{qx}=-\frac{qu}{qx}$.

(13). $\frac{q(au)}{qx}=\frac{qu}{qx}$. (14). $\frac{q(u^n)}{qx}=n\frac{qu}{qx}$.

(15). $$\frac{q(u^v)}{qx}=\left(\frac{q(u^v)}{qu}\right)\frac{qu}{qx}+\left(\frac{q(u^v)}{qv}\right)\frac{qv}{qx}=v\left(\frac{qu}{qx}+\log u\frac{qv}{qx}\right),$$

where $\left(\frac{q\,(u^v)}{qu}\right)$ signifies that v is to be considered constant, while the function is quotientiated with respect to u.

$$(16).\quad \frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx},$$

in which u is a function of z, and z in turn a function of x.

$$(17).\quad \frac{qu}{qx} = \frac{1}{\frac{qx}{qu}}.$$

5.—QUOTIENTIATION NOT DEPENDENT ON DIFFERENTIATION.

While, for the sake of convenience, we have made use of the well-known rules for differentiation in deriving the above formulæ, it is not necessary to do so, but each quotientiation formula can be derived by an independent limiting process. In fact, since

$$\frac{qy}{qx} = \frac{x}{y} \cdot \frac{dy}{dx},$$

each of the processes of differentiation and quotientiation can be expressed in terms of the other. Having once established independently the rules for quotientiation, all differentiation formulæ may be deduced from them. Some differentiation formulæ are thus more easily derived than by the ordinary methods. We proceed to derive a few of the leading quotientiation formulæ without the aid of differentiation.

(a).—*Quotiential Coefficient of ax^n.*

By definition

$$\frac{q\,(ax^n)}{qx} = \lim_{h=1} \left\{ \log_h \left[\frac{ax^n h^n}{ax^n} \right] \right\} = \lim_{h=1} \left\{ \log_h h^n \right\} = \lim_{h=1} (n) = n.$$

(b). *Quotiential Coefficient of e^x.*

By definition

$$\frac{q\,(e^x)}{qx} = \lim_{h=1} \left\{ \log_h \left(\frac{e^{xh}}{e^x} \right) \right\} = \lim_{h=1} \left\{ \frac{xh - x}{\log h} \right\} = x \cdot \lim_{h=1} \left\{ \frac{h-1}{\log h} \right\}.$$

Put $h-1=h'$, then

$$\frac{h-1}{\log h}=\frac{h'}{\log(1+h')}=\frac{h'}{h'-\frac{h'^2}{2}+\frac{h'^3}{3}-\text{etc.}}$$

and
$$\lim_{h=1}\left\{\frac{h-1}{\log h}\right\}=\lim_{h'=0}\left\{\frac{1}{1-\frac{h'}{2}+\frac{h'^2}{3}-\text{etc.}}\right\}=1.^*$$

Therefore, $\dfrac{q\,(e^x)}{qx}=x$.

(c). *Quotientiation of the Trigonometric Functions.*

$$\frac{q\,(\sin x)}{qx}=\lim_{h=1}\left\{\log_h\left(\frac{\sin xh}{\sin x}\right)\right\}=\lim_{h=1}\left\{\frac{\log\left(\frac{\sin xh}{\sin x}\right)}{\log h}\right\}.$$

Put $h=1+m$, then

$$\lim_{h=1}\left\{\frac{\log\left(\frac{\sin xh}{\sin x}\right)}{\log h}\right\}=\lim_{m=0}\left\{\frac{\log\frac{\sin x\cos mx+\cos x\sin mx}{\sin x}}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{\log(\cos mx+\cot x\sin mx)}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{\log(1+mx\cot x)}{\log(1+m)}\right\}$$

$$=\lim_{m=0}\left\{\frac{mx\cot x-\frac{(mx\cot x)^2}{2}+\text{etc.}}{m-\frac{m^2}{2}+\text{etc.}}\right\}=x\cot x,$$

that is,
$$\frac{q\,(\sin x)}{qx}=x\cot x.$$

(d). *Quotiential Coefficient of a Function of a Function.*

Let $u=f(z)$ and $z=\phi(x)$, and suppose that the substitution of $\phi(x)$ for z in $f(z)$ gives $u=F(x)$. Let us change x into xh, in consequence of which z becomes zk and u becomes ul, thus, $zk=\phi(xh)$, $ul=f(zk)$; also $ul=F(xh)$,

* The development of $\log(1+h)$, and its convergence is readily established without the aid of the differential calculus.

since x and u have obviously the same values in the third equation as in the first two.

Now, by definition, we have

$$\frac{qu}{qz} = \lim_{k=1}\left\{\log_k\left[\frac{f(zk)}{f(z)}\right]\right\} = \lim_{k=1}\left\{\log_k\left(\frac{ul}{u}\right)\right\} = \lim_{k=1}(\log_k l),$$

$$\frac{qz}{qx} = \lim_{h=1}\left\{\log_h\left[\frac{\phi(xh)}{\phi(x)}\right]\right\} = \lim_{h=1}\left\{\log_h\left(\frac{zk}{z}\right)\right\} = \lim_{h=1}(\log_h k).$$

Multiplying these two equations, remembering that k, h, and l approach unity together, we have

$$\frac{qu}{qz}\cdot\frac{qz}{qx} = \lim_{h=1}\{\log_k l\,.\log_h k\} = \lim_{h=1}\left\{\frac{\log l}{\log k}\cdot\frac{\log k}{\log h}\right\} = \lim_{h=1}\{\log_h l\}$$

$$= \lim_{h=1}\left\{\log_h\left(\frac{ul}{u}\right)\right\} = \lim_{h=1}\left\{\log_h\left[\frac{F(xh)}{F(x)}\right]\right\} = \frac{qu}{qx}\,.$$

6.—PARALLELISMS BETWEEN QUOTIENTIAL AND DIFFERENTIAL PROCESSES.

We propose now to trace some of the more important parallelisms which exist between quotiential and differential processes. In most cases, a mere inspection of the formulæ in §4 will justify the theorems as announced, in others what proof is necessary will be added. We shall use the abbreviations q. c. for quotiential coefficient and d. c. for differential coefficient.

(a). *Both processes are distributive, the one over a product, the other over a sum of functions,*

$$\frac{q(u_1\,.\,u_2\,\ldots\,u_n)}{qx} = \sum_{i=1}^{n}\frac{qu_i}{qx}\,, \qquad \frac{d(u_1 + u_2 + \ldots + u_n)}{dx} = \sum_{i=1}^{n}\frac{du_i}{dx}\,,$$

$u_1, u_2, \ldots, u_n$ being each a function of x.

If $(u_i = u_k)_{i,\,k,\,=1,\,2,\,\ldots\,n}$, there follow immediately the corollaries:

The q. c. of a function powered by a constant, is equal to that function multiplied by the q. c. of the variable,

$$\frac{q(u^n)}{qx} = n\,\frac{qu}{qx}\,,$$

$$\frac{q(x^n)}{qx} = n\,.$$

The d. c. of a function multiplied by a constant, is equal to that constant multiplied by the d. c. of the function,

$$\frac{d(nu)}{dx} = n\,\frac{du}{dx}\,,$$

$$\frac{d(nx)}{dx} = n\,.$$

(b). *The result of the quotientiation is not changed if the function is multiplied or divided by a constant,*

$$\frac{q\,(u \dot{\times} a)}{qx} = \frac{qu}{qx}.$$

The result of differentiation is not changed if the function is increased or diminished by a constant,

$$\frac{d\,(u \pm a)}{dx} = \frac{du}{dx}.$$

(c). *The q. c. of the reciprocal of a function is equal to the q. c. of the function with its sign changed,*

$$\frac{q\left(\frac{1}{u}\right)}{qx} = -\frac{qu}{qx}.$$

The d. c. of the negative of a function is equal to the d. c. of the function with its sign changed,

$$\frac{d\,(-u)}{dx} = -\frac{du}{dx}.$$

(d).

$$\frac{q\,(u_1 + u_2 + \ldots\ldots + u_n)}{qx} = \frac{1}{F}\sum_{i=1}^{n} u_i \frac{qu_i}{qx},$$

where $F = u_1 + u_2 + \ldots\ldots + u_n$.

$$\frac{d\,(u_1 . u_2 \ldots\ldots u_n)}{dx} = \sum_{i=1}^{n} u_i \frac{du_i}{dx}.$$

(e). $y = \log x$, when quotientiated becomes its reciprocal, $y = \dfrac{-1}{\log x}$ remains unchanged.

$y = e^x$, when differentiated, remains unchanged, $y = \sqrt{2x}$, becomes its reciprocal.

(f). *The q. c. of a function of a product of two independent variables is the same, whether we quotientiate with respect to one or the other of the variables,*

$$\frac{qf\,(xy)}{qx} = \frac{qf\,(xy)}{qy}.$$

The d. c. of a function of the sum of two independent variables is the same whether we differentiate with respect to one or the other of the variables,

$$\frac{df\,(x+y)}{dx} = \frac{df\,(x+y)}{dy}.$$

This theorem may be proven as follows:

$$\frac{qf\,(xy)}{qx} = \lim_{h=1}\left\{\log_h \frac{f\,(xh\,.\,y)}{f\,(xy)}\right\}$$

$$= \lim_{h=1}\left\{\log_h \frac{f\,(x\,.\,yh)}{f\,(xy)}\right\} = \frac{qf\,(xy)}{qy}.$$

(g). *If $u = F(z)$, $z = f(x)$, we have*

$$\frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx}. \qquad\qquad \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}.$$

(h). *If u and v are both functions of x, we have*

$$\frac{q(u^v)}{qx} = \left(\frac{q(u^v)}{qu}\right)\frac{qu}{qx} + \left(\frac{q(u^v)}{qv}\right)\frac{qv}{qx}, \qquad \frac{d(u^v)}{dx} = \frac{\partial(u^v)}{\partial u} \cdot \frac{du}{dx} + \frac{\partial(u^v)}{\partial v} \cdot \frac{dv}{dx},$$

and, generally, if F is a function of $u_1, u_2, \ldots, u_n$, where $u_1, u_2, \ldots, u_n$, are each functions of x, we have

$$\frac{qF}{qx} = \sum_{i=1}^{n}\left(\frac{qF}{qu_i}\right)\frac{qu_i}{qx}. \qquad\qquad \frac{dF}{dx} = \sum_{i=1}^{n}\frac{\partial F}{\partial u_i} \cdot \frac{du_i}{dx}.$$

(i). *Implicit functions.*

If in the last equation $F = 0$, it can readily be shown that

$$\left(\frac{qF}{qu_1}\right)\frac{qu_1}{qx} + \left(\frac{qF}{qu_2}\right)\frac{qu_2}{qx} + \cdots\cdots\cdots\left(\frac{qF}{qu_n}\right)\frac{qu_n}{qx} = 0.$$

For $F = F(x, y) = 0$, the equation becomes $\left(\frac{qF}{qx}\right) + \left(\frac{qF}{qy}\right)\frac{qy}{qx} = 0$, and hence

$$\frac{qy}{qx} = -\frac{\left(\frac{qF}{qx}\right)}{\left(\frac{qF}{qy}\right)}.$$

(j). *Successive Quotientiation.*

In general, the q. c. of a function of x will be again a function of x, and the q. c. of this function may be found by the preceding rules. We may, following the nomenclature of the differential calculus, designate this q. c. as the second quotiential coefficient of the original function, and write

$$\frac{q^2y}{qx^2} \quad \text{for} \quad \frac{q\left(\frac{qy}{qx}\right)}{qx},$$

and generally

$$\frac{q^ny}{qx^n} \quad \text{for} \quad \left(\frac{q}{qx}\right)^n y,$$

just as we write

$$\frac{d^ny}{dx^n} \quad \text{for} \quad \left(\frac{d}{dx}\right)^n y.$$

(k). *Change of independent variable.*

Since
$$\frac{qy}{qx}=\frac{1}{\frac{qx}{qy}},\quad \frac{qy}{qx}=\frac{qy}{qz}\cdot\frac{qz}{qx}=\frac{qy}{qz}\bigg/\frac{qx}{qz},$$

and hence
$$\frac{q^2y}{qx^2}=\left(\frac{q^2y}{qz^2}-\frac{q^2x}{qz^2}\right)\bigg/\frac{qx}{qz},$$
$$\frac{q^3y}{qx^3}=\left[\frac{\frac{q^2y}{qz^2}\cdot\frac{q^3y}{qz^3}-\frac{q^2x}{qz^2}\cdot\frac{q^3x}{qz^3}}{\frac{q^2y}{qz^2}-\frac{q^2x}{qz^2}}-\frac{q^2x}{qz^2}\right]\bigg/\frac{qx}{qz},$$

etc. etc. formulæ which must be used, when we wish to change the independent variable from x to z. When $z=y$, these formulæ become

$$\frac{qy}{qx}=1\bigg/\frac{qx}{qy},\quad \frac{q^2y}{qx^2}=-\frac{q^2x}{qy^2}\bigg/\frac{qx}{qy},$$
$$\frac{q^3y}{qx^3}=\left(\frac{q^3x}{qy^3}-\frac{q^2x}{qy^2}\right)\bigg/\frac{qx}{qy},\quad \text{etc. etc.}$$

7.—FUNCTIONS OF TWO INDEPENDENT VARIABLES.

THEOREM I.—*If $u=f(x+y)$, where x and y are independent variables, then*

$$\frac{qu}{qx}:\frac{qu}{qy}=x:y.$$

For, if we put $x+y=z$, we have

$$\frac{qu}{qx}=\frac{qu}{qz}\cdot\frac{qz}{qx},\quad\text{and}\quad\frac{qu}{qy}=\frac{qu}{qz}\cdot\frac{qz}{qy};$$

but
$$\frac{qz}{qx}=\frac{x}{x+y},\quad\text{and}\quad\frac{qz}{qy}=\frac{y}{x+y}.$$

Hence
$$\frac{qu}{qx}:\frac{qu}{qy}=\frac{qu}{qz}\cdot\frac{x}{x+y}:\frac{qu}{qz}\cdot\frac{y}{x+y}=x:y.$$

THEOREM II.—*If $u=f(xy)$, where x and y are independent variables, then*

$$\frac{q^nu}{qx^n}=\frac{q^nu}{qy^n}.$$

Let us assume that the theorem holds for some index r, we will show that it then holds for the next higher index $r+1$

Put
$$\frac{q^r u}{qx^r} = \frac{q^r u}{qy^r} = v, \quad \text{and} \quad xy = z,$$
then
$$\frac{q^{r+1}u}{qx^{r+1}} = \frac{qv}{qx} = \frac{qv}{qz} \cdot \frac{qz}{qx},$$
and
$$\frac{q^{r+1}u}{qy^{r+1}} = \frac{qv}{qy} = \frac{qv}{qz} \cdot \frac{qz}{qy},$$
but
$$\frac{qz}{qx} = 1, \quad \text{and} \quad \frac{qz}{qy} = 1,$$
hence
$$\frac{q^{r+1}u}{qx^{r+1}} = \frac{q^{r+1}u}{qy^{r+1}}.$$

Now the theorem has already been proven when $r = 1$ (*f.* §6.), hence it holds when $r = 2, 3, 4$, or any number.

THEOREM III.—*If* $u = f(x^y)$, *x and y being independent, then*
$$\frac{qu}{qx} : \frac{qu}{qy} = 1 : \log x.$$

Put $z = x^y$, then $\frac{qz}{qx} = y$, and $\frac{qz}{qy} = y \log x$;

also
$$\frac{qu}{qx} = \frac{qu}{qz} \cdot \frac{qz}{qx} = y \cdot \frac{qu}{qz}, \quad \frac{qu}{qy} = \frac{qu}{qz} \cdot \frac{qz}{qy} = y \log x \frac{qu}{qz}.$$
Hence,
$$\frac{qu}{qx} : \frac{qu}{qy} = y \frac{qu}{qz} : y \log x \frac{qu}{qz} = 1 : \log x.$$

If we put $y = \log y'$, that is, if $u = f(x^{\log y'})$, the theorem assumes the symmetric form $\frac{qu}{qx} : \frac{qu}{qy'} = \log y' : \log x$.

THEOREM IV.—*If* $u = f(x, y)$, *x and y being independent, then*
$$\frac{q^2u}{qxqy} \cdot \frac{qu}{qy} = \frac{q^2u}{qyqx} \cdot \frac{qu}{qx}.$$

Proof:
$$\frac{q^2u}{qxqy} = \frac{q\left(\frac{y}{u} \cdot \frac{du}{dy}\right)}{qx} = \frac{x\left(\frac{d^2u}{dxdy} - \frac{1}{u} \cdot \frac{du}{dx} \cdot \frac{du}{dy}\right)}{\frac{du}{dy}},$$
$$\frac{q^2u}{qyqx} = \frac{q\left(\frac{x}{u} \cdot \frac{du}{dx}\right)}{qy} = \frac{y\left(\frac{d^2u}{dydx} - \frac{1}{u} \cdot \frac{du}{dy} \cdot \frac{du}{dx}\right)}{\frac{du}{dx}},$$

whence $$\frac{q^2u}{qxqy} : \frac{q^2u}{qyqx} = \frac{x}{\frac{du}{dy}} : \frac{y}{\frac{du}{dx}} = \frac{x}{u} \cdot \frac{du}{dx} : \frac{y}{u} \cdot \frac{du}{dy} = \frac{qu}{qx} : \frac{qu}{qy},$$

or $$\frac{q^2u}{qxqy} \cdot \frac{qu}{qy} = \frac{q^2u}{qyqx} \cdot \frac{qu}{qx}.$$

Cor. 1. *If* $u = f(x+y)$, *then* $\frac{q^2u}{qx\,qy} \cdot y = \frac{q^2u}{qy\,qx} \cdot x$, for in that case it has been shown in theorem I, $\frac{qu}{qx} : \frac{qu}{qy} = x : y$.

Cor. 2. In order that $\frac{q^2u}{qx\,qy} = \frac{q^2u}{qy\,qx}$, we must have $\frac{qu}{qx} = \frac{qu}{qy}$, hence, *the necessary and sufficient condition that the same result be obtained whether a function be quotientiated first with respect to x and then with respect to y, or first with respect to y and then with respect to x, is that u satisfies the relation* $\frac{qu}{qx} = \frac{qu}{qy}$. $u = f(xy)$ evidently satisfies this criterion.

8.—FUNCTIONS OF THREE OR MORE INDEPENDENT VARIABLES.

Let, now, $u = f(x, y, z)$, where x, y and z are independent variables, we then have, by theorem IV of the last paragraph,

$$\frac{q^2u}{qy\,qz} = \frac{q^2u}{qzqy} \cdot \frac{qu}{qy} \Big/ \frac{qu}{qz},$$

and hence

$$\frac{q^3u}{qx\,qy\,qz} = \frac{q\left(\frac{q^2u}{qyqz}\right)}{qx} = \frac{q^3u}{qxqzqy} + \frac{q^2u}{qxqy} - \frac{q^2u}{qx\,qz},$$

or $$\frac{q^3u}{qx\,qy\,qz} + \frac{q^2u}{qx\,qz} = \frac{q^3u}{qxqzqy} + \frac{q^2u}{qxqy}. \qquad [A]$$

Again, putting $v = \frac{qu}{qz}$, we have

$$\frac{q^3u}{qx\,qy\,qz} = \frac{q^2v}{qxqy} = \frac{q^2v}{qyqx} \cdot \frac{qv}{qx} \Big/ \frac{qv}{qy} = \frac{q^3u}{qyqxqz} \cdot \frac{q^2u}{qx\,qz} \Big/ \frac{q^2u}{qyqz},$$

or $$\frac{q^3u}{qxqy\,qz} \cdot \frac{q^2u}{qy\,qz} = \frac{q^3u}{qyqxqz} \cdot \frac{q^2u}{qxqz}. \qquad [B]$$

Finally, multiply [A] by $\frac{q^2u}{qz\,qy}$,

$$\frac{q^3u}{qxqyqz}\cdot\frac{q^2x}{qzqy}+\frac{q^2u}{qxqz}\cdot\frac{q^2u}{qzqy}=\frac{q^3u}{qxqzqy}\cdot\frac{q^2u}{qzqy}+\frac{q^2u}{qxqy}\cdot\frac{q^2u}{qzqy},$$

from [B],

$$\frac{q^3u}{qxqzqy}\cdot\frac{q^2u}{qzqy}=\frac{q^3u}{qzqxqy}\cdot\frac{q^2u}{qxqy},$$

and interchanging x and z in [A] and multiplying by $\frac{q^2u}{qx\,qy}$,

$$\frac{q^3u}{qzqxqy}\cdot\frac{q^2u}{qxqy}+\frac{q^2u}{qzqy}\cdot\frac{q^2u}{qxqy}=\frac{q^3u}{qzqyqx}\cdot\frac{q^2u}{qxqy}+\frac{q^2u}{qzqx}\cdot\frac{q^2u}{qxqy}.$$

Adding the last three equations and simplifying the result we obtain

$$\left[\frac{q^3u}{qxqyqz}+\frac{q^2u}{qxqz}+\frac{q^2u}{qxqy}\right]\frac{q^2u}{qzqy}=\left[\frac{q^3u}{qzqyqx}+\frac{q^2u}{qzqx}+\frac{q^2u}{qzqy}\right]\frac{q^2u}{qxqy}. \qquad \text{[C]}$$

Formulæ [A], [B], [C], give the rules according to which we may interchange in any order whatsoever the quotientiation with respect to the three independent variables x, y, z. The law of the interchange of x and y in $\frac{q^3u}{qxqy^2}$ is obtained from [C] by putting $z=y$. We get

$$\left[\frac{q^3u}{qxqy^2}+\frac{q^2u}{qxqy}\right]\frac{q^2u}{qy^2}=\left[\frac{q^3u}{qy^2qx}+\frac{q^2u}{qyqx}\right]\frac{q^2u}{qxqy}.$$

We next consider the general problem: *Given a function of n variables $u=f(x_1, x_2, \ldots, x_n)$; required the rule by which the n^{th} q.c. of u with respect to the n variables $x_1, x_2, \ldots, x_n$, may be replaced by the n^{th} q.c. of u with respect to the same variables in the same order except that the quotientiation with respect to any two particular variables as x_i and x_k is interchanged.*

We shall find it convenient to use an abridged notation so we shall write

Q_xu for $\frac{qu}{qx}$, q_x^2u for $\frac{q^2u}{qx^2}$, similarly $Q^2_{x_1x_2}u\equiv\frac{q^2u}{qx_1qx_2}$, $Q^3_{x_1x_2x_3}u\equiv\frac{q^3u}{qx_1qx_2qx_3}$, etc.;

and finally, when there is no danger of confusion, we put Qu for Q_xu, Q^2u for Q^2_xu, $Q_{12}u$ for $Q^2_{x_1x_2}u$, $Q_{123}u$ for $Q^3_{x_1x_2x_3}u$, etc. Using this notation, the formulæ of theorem IV, §7. and [A], [B], [C] of this paragraph, become respectively

$$Q_{12}u \,.\, Q_2u = Q_{21}u \,.\, Q_1u,$$
$$Q_{123}u + Q_{13}u = Q_{132}u + Q_{12}u,$$
$$Q_{123}u \,.\, Q_{23}u = Q_{213}u \,.\, Q_{13}u,$$
$$[Q_{123}u + Q_{13}u + Q_{12}u]\, Q_{32}\, u = [Q_{321}u + Q_{31}u + Q_{32}u]\, Q_{12}u.$$

Let us now suppose that the laws for the interchange of variables in the q. c.'s of the n^{th} and lower orders are known, and that the equations expressing these laws are rational and linear in the q. c. of the highest order. Let $Q^{n+1}_{1, 2, \ldots, i, \ldots, k, \ldots, n+1}u$ be the q. c. in question, and let it be required to find the law for the interchange of x_i and x_k. Then three cases arise:

Case I.—When $i, k \neq 1$.

Under this assumption, we put

$$Q^{n+1}_{1, 2, \ldots, i, \ldots, k, \ldots, n+1}u = Q_1 Q^n_{2, \ldots, i, \ldots, k, \ldots, n+1}u.$$

Now, the law for the interchange of variables in $Q^n_{2, \ldots, i, \ldots, k, \ldots, n+1}u$ is known, hence, if we quotientiate with respect to x the equation which expresses this law, we get an equation which is seen to involve only terms of the first degree in $Q^{n+1}_{1, 2, \ldots, i, \ldots, k, \ldots, n+1}u$ and $Q^{n+1}_{1, 2, \ldots, k, \ldots, i, \ldots, n+1}u$, together with q. c.'s of lower orders, i. e. we get the law desired.

Case II.—When $i = 1,\ k \neq n + 1$.

Put $Q^{n-k+1}_{k+1, \ldots, n+1}u = v$, then $Q^{n+1}_{i, 2, \ldots, k, \ldots, n+1}u = Q^k_{i, 2, \ldots, k}v$. The relation between $Q^k_{i, 2, \ldots, k}v$ and $Q^k_{k, 2, \ldots, i}v$ is known, k being at most equal to n, hence is also the relation between $Q^{n+1}_{i, 2, \ldots, k, \ldots, n+1}u$ and $Q^{n+1}_{k, 2, \ldots, i, \ldots, n+1}u$.

Case III.—When $i = 1,\ k = n + 1$.

Consider any third variable as x_j. We first establish the law for the interchange of x_j and x_k by case I. This gives $Q^{n+1}_{i, 2, \ldots, j, \ldots, k}u$ in terms of $Q^{n+1}_{i, 2, \ldots, k, \ldots, j}u$. Then we express $Q^{n+1}_{i, 2, \ldots, k, \ldots, j}u$ in terms of $Q^{n+1}_{k, 2, \ldots, i, \ldots, j}u$ by case II. Applying now case I again we express $Q^{n+1}_{k, 2, \ldots, i, \ldots, j}u$ in terms of $Q^{n+1}_{k, 2, \ldots, j, \ldots, i}u$. Between the equations expressing these relations we eliminate $Q^{n+1}_{i, 2, \ldots, k, \ldots, j}u$ and $Q^{n+1}_{k, 2, \ldots, i, \ldots, j}u$ and obtain an equation between $Q^{n+1}_{i, 2, \ldots, j, \ldots, k}u$, $Q^{n+1}_{k, 2, \ldots, j, \ldots, i}u$ and q. c's of a lower order than $n + 1$. Moreover this equation is rational, and linear in the q. c's of the $(n + 1)^{\text{th}}$ order.

We conclude then, that knowing the law for the interchange of variables in a q. c. of one order we can deduce the law for the interchange of variables in

the q. c's of the next higher, and hence of any higher order, and having already established these laws for the q. c's of the second and third orders, the problem is theoretically solved. The general formula can not be compactly exhibited, but can be readily worked out for any special case. Applied to the possible cases of the various quotiential coefficients of the fourth order, we have

$$Q_{1234}\,u + Q_{134}\,u = Q_{1324}\,u + Q_{124}\,u,$$
$$Q_{1234}u \,.\, Q_{234}u = Q_{1234}u \,.\, Q_{134}u,$$
$$[Q_{1234}u + Q_{134}u]\; Q_{324}u = [Q_{3214}u + Q_{314}u]\; Q_{124}u,$$
$$Q_{1234}u \,.\, Q_{234}u + Q_{124}u \,.\, Q_{24}u = Q_{1243}u \,.\, Q_{243}u + Q_{123}u \,.\, Q_{23}u,$$
$$[Q_{1234}u + Q_{143}u]\; Q_{234}u + [Q_{124}u + Q_{143}u]\; Q_{24}u$$
$$= [Q_{1432}u + Q_{123}u]\; Q_{432}u + [Q_{142}u + Q_{123}u]\; Q_{42}u,$$
$$\{Q_{1234}u \,.\, Q_{234}u + Q_{124}u \,.\, Q_{24}u + [Q_{13}u + Q_{143}u]\; Q_{243}u\}\; Q_{423}u \,.\, Q_{213}u$$
$$= \{Q_{4231}u \,.\, Q_{231}u + Q_{421}u \,.\, Q_{21}u + [Q_{43}u + Q_{413}u]\; Q_{213}u\}\; Q_{123}u \,.\, Q_{243}u.$$

It will be observed that *the right member of each of the above equations can be written down from the corresponding left members, by simply interchanging the suffixes i and k of the variables x_i and x_k, and it can be shown that this is true generally.*

9.—SUCCESSIVE QUOTIENTIATION.

We now return to functions of a single variable x. As a rule, the n^{th} q. c. of $y = f(x)$ does not admit of a simple algebraic expression, but there is at least one exception. For $y = \log x$, $\dfrac{qy}{qx} = \dfrac{1}{\log x}$, $\dfrac{q^2y}{qx^2} = \dfrac{-1}{\log x}$, and after that, every successive q. c. is equal to $\dfrac{-1}{\log x}$, so that

$$\frac{q^n(\log x)}{qx^n} = \pm\frac{1}{\log x},$$

according as $n = 1$ or $n \neq 1$.

The second q. c. can in most cases be readily expressed. If $y = \log^n x$, we have

$$\frac{qy}{qx} = \frac{1}{\log^n x}\cdot\frac{1}{\log^{n-1} x}\,\cdots\,\frac{1}{\log x},$$

and

$$\frac{q^2y}{qx^2} = -\left(\frac{q\,\log^n x}{qx} + \frac{q\,\log^{n-1} x}{qx} + \cdots + \frac{q\,\log x}{qx}\right).$$

If $y = u \,.\, v \,.\, w \;\ldots\ldots,\; u,\, v,\, w,\; \ldots\ldots$ being functions of x,

$$\frac{qy}{qx} = \sum \frac{qu}{qx} \quad \text{and} \quad \frac{q^2y}{qx^2} = \sum \frac{qu}{qx}\,\frac{q^2u}{qx^2} \Big/ \sum \frac{qu}{qx}.$$

If $y = u + v + w + \;\ldots\ldots,$

$$\frac{qy}{qx} = \frac{\sum u \frac{qu}{qx}}{\sum u}, \quad \frac{q^2y}{qx^2} = \frac{\sum u \frac{q^2u}{qx^2} + \sum u \frac{qu}{qx}\,\frac{q^2u}{qx^2}}{\sum u \frac{qu}{qx}} - \frac{\sum u \frac{qu}{qx}}{\sum u}$$

Using the notation introduced in the preceding paragraph, these formulæ may be written:

For $y = u \,.\, v \,.\, w \;\ldots\ldots$

$$Qy = \Sigma\, Qu, \quad Qy \,.\, Q^2y = \Sigma\, Qu\, Q^2u;$$

For $y = u + v + w + \;\ldots\ldots$

$$y\,Qy = \Sigma u Qu, \quad y\,Qy\,Q^2y + y\,\overline{Qy}^{2} = \Sigma u\, Qu\, Q^2u + \Sigma u\, \overline{Qu}^{2}.$$

Let us find an equation involving the third q. c. of $y = u + v + w + \;\ldots\ldots$ To do this we quotientiate the expression

$$y\,Qy\,Q^2y + y\,\overline{Qy}^{2} = \Sigma u\, Qu\, Q^2u + \Sigma u\, \overline{Qu}^{2},$$

and obtain

$$\frac{y\,Qy\,Q^2y\,(Qy + Q^2y + Q^3y) + y\,\overline{Qy}^{2}\,(Qy + 2\,Q^2y)}{y\,Qy\,Q^2y + y\,\overline{Qy}^{2}}$$

$$= \frac{\Sigma u\, Qu\, Q^2u\,(Qu + Q^2u + Q^3u) + \Sigma u\,\overline{Qu}^{2}\,(Qu + 2\,Q^2u)}{\Sigma u\, Qu\, Q^2u + \Sigma u\, \overline{Qu}^{2}},$$

and cancelling the equal denominators and collecting like terms,

$$y\,\overline{Qy}^{3} + 3y\,\overline{Qy}^{2}\,Q^2y + y\,Qy\,\overline{Q^2y}^{2} + y\,Qy\,Q^2y\,Q^3y$$

$$= \sum u\,\overline{Qu}^{3} + 3\sum u\,\overline{Qu}^{2}\,Q^2u + \sum u\, Qu\,\overline{Q^2u}^{2} + \sum u\, Qu\, Q^2u\, Q^3u.$$

An examination of this expression involving Q^3y, and a comparison of it with the similar expressions involving Q^2y and Qy respectively, suggests the following law of formation:

(a). *The left member of the equation is homogeneous of the fourth degree in* $y,\ Qy,\ Q^2y,\ Q^3y$.

(b). *Each term contains the factor y and a cofactor of degree three, which is a multiple of Qy, and consecutive ones of the quantities, Qy, Q^2y, Q^3y.*

(c). *All the terms, 2^2 in number, that can thus be formed, occur.*

(d). *The sum of all the coefficients on the left is* 3 !

(e). *The right side of the equations can be formed from the left side by replacing every term on the left by a sum of terms formed by substituting in that term for y successively the quantities u, v, w, etc. and adding the results.*

We now show that these statements can be generalized for an expression involving Q^ny and quotiential coefficients of lower orders. To do this we assume the existence of an expression formed according to the above law involving $Q^{n-1}y$ and quotiential coefficients of lower orders. Quotientiating this gives an expression, involving Q^ny and lower quotientials, which obeys the same law. But we have shown the existence of such an expression when $n-1=3$, hence the law holds in general.

(a). *The terms on the left are homogeneous of the $n+1^{\text{th}}$ degree in y, Qy, Q^2y, $\ldots$, Q^ny, where n is the order of the highest quotiential which is involved.*

Notice, first, that since the q. c. of a sum of terms is equal to a sum of products formed by multiplying each term by its q. c., adding the results and dividing the sum of the partial products by the original sum, the divisors on the two sides of the equation, after quotientiation, will cancel, being equal. The quotientiation of a sum of terms equal to another sum reduces then to a consideration of products of the individual terms by their quotientials.

Let

$$Ay\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\ldots\overline{Q^sy}^{\sigma},\quad \alpha+\beta+\ldots+\sigma=n-1,\quad s\leqq n-1,$$

be a term of the assumed equation. Multiplying this by its q. c., we obtain

$$Ay\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\ldots\overline{Q^sy}^{\sigma}(Qy+\alpha Q^2y+\beta Q^3y+\ldots+\sigma Q^{s+1}y).$$

This shows that the degree of every term is raised by unity, and hence, if the assumed expression is homogeneous of degree n, the expression obtained by quotientiation must be homogeneous and of degree $n+1$. A new factor Q^ny is introduced in the case, $s=n-1$, so that the new expression involves y, Qy, Q^2y, $\ldots$, Q^ny.

(b). *Each term contains the factor y, and a cofactor of degree n, which is some multiple of Qy and consecutive ones of the quantities Qy, Q^2y, $\ldots$, Q^ny.*

That each term contains the factor y is obvious when we remember that each term of the result of the quotientiation contains some term of the original expression as a factor. Again, every term except the last arising from the quotientiation of a single term contains all the constituent factors of that term, while the last term contains, besides the original constituents Qy, Q^2y, ..., Q^sy, the new constituent $Q^{s+1}y$.

(c). *All the terms of degree* $n+1$, 2^{n-1} *in number, that can be formed to satisfy* (*b*), *occur.*

Every term of the $(n+1)^{\text{th}}$ degree can be formed from some term of the n^{th} degree, by either increasing the exponent of some Q by unity, or by adding the factor Q^ny. Both possibilities are exhausted during the process of quotientiation. Hence, all possible terms occur in the expression of the $(n+1)^{\text{th}}$ degree provided all possible terms exist in the assumed expression.

The total number of these terms is 2^{n-1}, for, to every term in the assumed expression as

$$y\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\,\ldots\,\overline{Q^sy}^{\sigma},\quad 1+\alpha+\beta+\ldots+\sigma=n,\ s\leqq n-1,$$

correspond two terms in the final expression, viz.

$$y\,Qy\,\overline{Q^2y}^{\alpha}\,\overline{Q^3y}^{\beta}\,\ldots\,\overline{Q^{s+1}y}^{\sigma},\quad 1+1+\alpha+\beta+\ldots+\sigma=n+1,\ s+1\leqq n,\qquad [A]$$

and

$$y\,\overline{Qy}^{\alpha+1}\,\overline{Q^2y}^{\beta}\,\ldots\,\overline{Q^sy}^{\sigma},\quad 1+(\alpha+1)+\beta+\ldots+\sigma=n+1,\ s<n,\qquad [B]$$

It is clear that the terms thus derived are all different. That the totality of these pairs of terms exhausts all possible terms in the final expression is evident, since any term in the final expression as

$$y\,\overline{Qy}^{\alpha'}\,\overline{Q^2y}^{\beta'}\,\ldots\,\overline{Q^ry}^{\sigma'},\quad 1+\alpha'+\beta'+\ldots+\sigma'=n+1,\quad r\leqq n,$$

is of the form [A] or [B] according as $\alpha=1$ or $\alpha\neq 1$. Finally, because of this one-to-two correspondence between the 2^{n-2} terms of the assumed expression, and those of the result of quotientiation, the number of terms in the latter is $2^{n-2}.\,2$ or 2^{n-1}.

(d). *The sum of all the coefficients on the left is* $n!$

Let one term of the original expression be $Ay\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\,\ldots\,\overline{Q^sy}^{\sigma}$. Quotientiating this, we get, for corresponding term in the numerator,

$$Ay\,\overline{Qy}^{\alpha}\,\overline{Q^2y}^{\beta}\,\ldots\,\overline{Q^sy}^{\sigma}\,(Qy+\alpha Q^2y+\beta Q^3y+\ldots+\sigma Q^{s+1}y),$$

the sum of the coefficients of which is $A(1+\alpha+\beta+\ldots\ldots+\sigma)=An$. Similarly for every other term. The original coefficient sum $n-1\,!$ is hence multiplied by n, giving $n!$ for the final coefficient sum.

(e). *The right side of the equation can be formed from the left side by replacing every term on the left by a sum of terms, formed by substituting in that term for y successively the quantities u, v, w, etc., and adding the results.*

To any one term in the assumed expression, as $Ay\,\overline{Qy}^{\,\alpha}\,\overline{Q^2y}^{\,\beta}\ldots\ldots\overline{Q^sy}^{\,\sigma}$, there corresponds on the right-hand side a sum of terms $A\Sigma u\,\overline{Qu}^{\,\alpha}\,\overline{Q^2u}^{\,\beta}\ldots\ldots\overline{Q^su}^{\,\sigma}$. Consider the terms that are yielded by these corresponding terms on quotientiation. The left term yields

$$Ay\,\overline{Qy}^{\,\alpha+1}\,\overline{Q^2y}^{\,\beta}\ldots\ldots\overline{Q^sy}^{\,\sigma}+\alpha Ay\,\overline{Qy}^{\,\alpha}\,\overline{Q^ry}^{\,\beta+1}\ldots\ldots\overline{Q^sy}^{\,\sigma}+\ldots\ldots$$
$$+\,\sigma Ay\,\overline{Qy}^{\,\alpha}\,\overline{Q^2y}^{\,\beta}\ldots\ldots Q^{s+1}y,$$

while the corresponding right side sum yields

$$\begin{aligned}&Au\,\overline{Qu}^{\,\alpha}\,\overline{Q^2u}^{\,\beta}\ldots\ldots\overline{Q^su}^{\,\sigma}(Qu+\alpha Q^2u+\beta Q^3u+\ldots\ldots+\sigma Q^{s+1}u)\\&+Av\,\overline{Qv}^{\,\alpha}\,\overline{Q^rv}^{\,\beta}\ldots\ldots\overline{Q^sv}^{\,\sigma}(Qv+\alpha Q^2v+\beta Q^3v+\ldots\ldots+\sigma Q^{s+1}v)\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+\text{etc.}\end{aligned}$$

This, by summing terms which stand under each other, may be written

$$A\Sigma u\,\overline{Qu}^{\,\alpha+1}\,\overline{Q^2u}^{\,\beta}\ldots\ldots\overline{Q^su}^{\,\sigma}+\alpha A\Sigma u\,\overline{Qu}^{\,\alpha}\,\overline{Q^2u}^{\,\beta+1}\ldots\ldots\overline{Q^su}^{\,\sigma}+\ldots\ldots$$
$$+\,\sigma A\Sigma u\,\overline{Qu}^{\,\alpha}\,\overline{Q^2u}^{\,\beta}\ldots\ldots Q^{s+1}u.$$

We see then that the law expressed in (e) holds for the results of corresponding terms, hence also for the collected results.

We have now determined everything except the specific coefficients of the individual terms. To find these, let us consider the general term of the expression giving Q^ny, viz.

$$A_ny\,\overline{Qy}^{\,\alpha}\,\overline{Q^2y}^{\,\beta}\,\overline{Q^3y}^{\,\gamma}\ldots\ldots\overline{Q^ry}^{\,\rho}\,\overline{Q^sy}^{\,\sigma},\quad \alpha+\beta+\gamma+\ldots\ldots+\rho+\sigma=n.$$

This term is obtained by multiplying each of the following terms belonging to

the expression for $Q^{n-1}y$ by its respective q. c.,

$$\begin{array}{llll}
A_{n-1} y \overline{Qy}^{\alpha-1} \overline{Q^2y}^{\beta} \overline{Q^3y}^{\gamma} \cdots\cdots \overline{Q^ry}^{\rho} \overline{Q^sy}^{\sigma} & \text{by} & Qy, \\
B_{n-1} y \overline{Qy}^{\alpha} \overline{Q^2y}^{\beta-1} \overline{Q^3y}^{\gamma} \cdots\cdots \overline{Q^ry}^{\rho} \overline{Q^sy}^{\sigma} & `` & \alpha Q^2y, \\
C_{n-1} y \overline{Qy}^{\alpha} \overline{Q^2y}^{\beta} \overline{Q^3y}^{\gamma-1} \cdots\cdots \overline{Q^ry}^{\rho} \overline{Q^sy}^{\sigma} & `` & \beta Q^3y, \\
\vdots & & \vdots \\
S_{n-1} y \overline{Qy}^{\alpha} \overline{Q^2y}^{\beta} \overline{Q^3y}^{\gamma} \cdots\cdots \overline{Q^ry}^{\rho} \overline{Q^sy}^{\sigma-1} & `` & \rho Q^sy.
\end{array}$$

Adding these products and collecting the coefficients, we see that

$$A_n = A_{n-1} + \alpha B_{n-1} + \beta C_{n-1} + \cdots\cdots + \rho S_{n-1},$$

a formula which enables us to compute the coefficients of the expression for Q^ny from those of the expression for $Q^{n-1}y$. The computation of the complete expression for Q^ny becomes laborious when n is large. The properties set forth under a), b), c), d) and e) furnish efficient checks for the computation. Below we give the expressions for $n = 4$, 5 and 6.

$n = 4.$

$$\begin{aligned}
&y\overline{Qy}^{4} + 6y\overline{Qy}^{3}Q^2y + 7y\overline{Qy}^{2}\overline{Q^2y}^{2} + 4y\overline{Qy}^{2}Q^2yQ^3y + yQy\overline{Q^2y}^{3} + 3yQy\overline{Q^2y}^{2}Q^3y \\
&\qquad + yQyQy^2\overline{Q^3y}^{2} + yQyQ^2yQ^3yQ^4y = \Sigma u\overline{Qu}^{4} + etc.
\end{aligned}$$

$n = 5.$

$$\begin{aligned}
&y\overline{Qy}^{5} + 10y\overline{Qy}^{4}Q^2y + 25y\overline{Qy}^{3}\overline{Q^2y}^{2} + 10y\overline{Qy}^{3}Q^2yQ^3y + 15yQy\overline{Q^2y}^{3} + 25y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2} \\
&\qquad + 5y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2} + 5y\overline{Qy}^{2}Q^2yQ^3yQ^4y + yQy\overline{Q^2y}^{4} + 6yQy\overline{Q^2y}^{3}Q^3y \\
&\qquad + 7yQy\overline{Q^2y}^{2}\overline{Q^3y}^{2} + 4yQy\overline{Q^2y}^{2}Q^3yQ^4y + yQyQ^2y\overline{Q^3y}^{3} \\
&\qquad + 3yQyQ^2y\overline{Q^3y}^{2}Q^4y + yQyQ^2yQ^3y\overline{Q^4y}^{2} + yQyQ^2yQ^3yQ^4yQ^5y \\
&\qquad = \Sigma u\overline{Qu}^{5} + etc.
\end{aligned}$$

$n = 6.$

$$
\begin{aligned}
&y\overline{Qy}^{6} + 15y\overline{Qy}^{5}Q^2y + 65y\overline{Qy}^{4}\overline{Q^2y}^{2} + 20y\overline{Qy}^{4}Q^2yQ^3y + 90\,y\overline{Qy}^{3}\overline{Q^2y}^{3} \\
&\quad + 105y\overline{Qy}^{3}\overline{Q^2y}^{2}Q^3y + 15y\overline{Qy}^{3}Q^2y\overline{Q^3y}^{2} + 15y\overline{Qy}^{3}Q^2yQ^3yQ^4y + 31y\overline{Qy}^{2}\overline{Q^2y}^{4} \\
&\quad + 101y\overline{Qy}^{2}\overline{Q^2y}^{3}Q^3y + 67y\overline{Qy}^{2}\overline{Q^2y}^{2}\overline{Q^3y}^{2} + 39y\overline{Qy}^{2}\overline{Q^2y}^{2}Q^3yQ^4y \\
&\quad + 6y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{3} + 18y\overline{Qy}^{2}Q^2y\overline{Q^3y}^{2}Q^4y + 6y\overline{Qy}^{2}Q^2yQ^3y\overline{Q^4y}^{2} \\
&\quad + 6y\overline{Qy}^{2}Q^2yQ^3yQ^4yQ^5y + yQy\overline{Q^2y}^{5} + 10yQy\overline{Q^2y}^{4}Q^3y + 25yQy\overline{Q^2y}^{3}\overline{Q^3y}^{2} \\
&\quad + 10yQy\overline{Q^2y}^{3}Q^3yQ^4y + 15yQy\overline{Q^2y}^{2}\overline{Q^3y}^{3} + 25yQy\overline{Q^2y}^{2}\overline{Q^3y}^{2}Q^4y \\
&\quad + 5yQy\overline{Q^2y}^{2}Q^3y\overline{Q^4y}^{2} + 5yQy\overline{Q^2y}^{2}Q^3yQ^4yQ^5y + yQyQ^2y\overline{Q^3y}^{4} \\
&\quad + 6yQyQ^2y\overline{Q^3y}^{3}Q^4y + 7yQyQ^2y\overline{Q^3y}^{2}\overline{Q^4y}^{2} + 4yQyQ^2y\overline{Q^3y}^{2}Q^4yQ^5y \\
&\quad + yQyQ^2yQ^3y\overline{Q^4y}^{3} + 3yQyQ^2yQ^3y\overline{Q^4y}^{2}Q^5y + yQyQ^2yQ^3yQ^4y\overline{Q^5y}^{2} \\
&\quad + yQyQ^2yQ^3yQ^4yQ^5yQ^6y = \Sigma u\overline{Qu}^{6} + \textit{etc}.
\end{aligned}
$$

It is now easy to find the successive q. c's of a product. For if

$$y = u\,.\,v\,.\,w\,\ldots\,,$$

then

$$Qy = Qu + Qv + Qw + \ldots\,,$$

so that, in order to form the expression for the n^{th} q. c. of a product, we need only to substitute in the expression for the $n - 1^{\text{th}}$ q. c. of a sum

$$Qy,\ Qu,\ Qv,\text{ etc.,}\quad\text{for}\quad y,\ u,\ v,\text{ etc.,}$$

and generally

$$Q^{r+1}y,\ Q^{r+1}u,\ Q^{r+1}v,\text{ etc.,}\quad\text{for}\quad Q^ry,\ Q^ru,\ Q^rv,\text{ etc.}$$

Thus, if $y = u\,.\,v\,.\,w\,\ldots$, we have for $n = 3$ and $n = 4$ respectively

$$Qy\,\overline{Q^2y}^{3} + 3\,Qy\,\overline{Q^2y}^{2}\,Q^3y + Qy\,Q^2y\,\overline{Q^3y}^{2} + Qy\,Q^2y\,Q^3y\,Q^4y = \Sigma\,Qu\,\overline{Q^2u}^{3} + \text{etc.},$$

$$
\begin{aligned}
&Qy\,\overline{Q^2y}^{4} + 6\,Qy\,\overline{Q^2y}^{3}\,Q^3y + 7\,Qy\,\overline{Q^2y}^{2}\,\overline{Q^3y}^{2} + 4\,Qy\,\overline{Q^2y}^{2}\,Q^3y\,Q^4y + Qy\,Q^2y\,\overline{Q^3y}^{3} \\
&\quad + 3\,Qy\,Q^2y\,\overline{Q^3y}^{2}\,Q^4y + Qy\,Q^2y\,Q^3y\,\overline{Q^4y}^{2} + Qy\,Q^2y\,Q^3y\,Q^4y\,Q^5y = \Sigma\,Qu\,\overline{Q^2u}^{4} + \text{etc.},
\end{aligned}
$$

10.—SYZYGIES CONNECTING DIFFERENTIAL AND QUOTIENTIAL PROCESSES.

It remains to consider operations into which differential and quotiential processes enter conjointly. In discussing such, it is convenient to use the follow-

ing notation:

$$Dy = \frac{dy}{dx}, \quad QDy = \frac{q(Dy)}{qx}, \quad DQy = \frac{d(Qy)}{dx},$$

similarly,

$$DQDy = \frac{d(QDy)}{dx}, \quad D^2Qy = \frac{d(DQy)}{dx}, \quad QDQy = \frac{q(DQy)}{qx}, \text{ etc.}$$

It is now easy to express these various mixed operations in terms of successive q. c's alone. We have

I. 1. $Dy = \frac{y}{x} Qy,$

II. 2. $QDy = Q\left(\frac{y}{x} Qy\right) = Q^2y + Qy - 1,$

similarly,

3. $DQy = \frac{1}{x} Qy Q^2y,$

4. $D^2y = \frac{y}{x^2} Qy(Q^2y + Qy - 1),$

III. 5. $Q^2Dy = \frac{Q^2y(Q^3y + Qy)}{Q^2y + Qy - 1},$

6. $QDQy = Q^3y + Q^2y - 1,$

7. $QD^2y = \frac{Q^2y(Q^3y + Qy) - 1}{Q^2y + Qy - 1} + 1,$

8. $DQ^2y = \frac{Q^2yQ^3y}{x},$

9. $DQDy = \frac{Q^2y}{x}(Q^3y + Qy),$

10. $D^2Qy = \frac{QyQ^2y}{x^2}(Q^3y + Q^2y - 1),$

11. $D^3y = \frac{y}{x^3} Qy(\overline{QDy}^2 - QDy + xDQDy)$

$$= \frac{y}{x^3} Qy[(Q^2y + Qy - 1)^2 - (Q^2y + Qy - 1) + (Q^3yQ^2y + Q^2yQy)].$$

These mixed operations may also be expressed in terms of differential processes alone, but in most cases the resulting formulæ are more complex than

those given above. For instance, 6) and 8) become respectively

$$QDQy = \frac{x}{y} \cdot \frac{2x\overline{Dy}^{3} - 2y\overline{Dy}^{2} - 3xyDyD^2y + 2y^2D^2y + xy^2D^3y}{yDy + xyD^2y - x\overline{Dy}^{2}},$$

$$DQ^2y = \frac{x\overline{Dy}^{4} - y\overline{Dy}^{3} - xy\overline{Dy}^{2}D^2y + y^2DyD^2y + xy^2DyD^3y - xy^2\overline{D^2y}^{2}}{y^2\overline{Dy}^{2}}.$$

Each of the groups of operations, I, II, III, is connected by a remarkable syzygy, which affords a convenient check in the computation of successive groups.

I. We may write

$$\frac{Dy}{Qy} \cdot \frac{x}{y} = 1.$$

If we eliminate Qy and $\overline{Qy}^{2}$ from the equations under II, we have

$$\frac{D^2y}{QDy} \cdot \frac{Q^2y}{DQy} \cdot \frac{x}{y} = 1,$$

Eliminating Qy and Q^2y from the equations under III, we find

$$\frac{D^3y}{Q^3y} \cdot \frac{Q^2Dy}{D^2Qy} \cdot \frac{DQ^2y}{QD^2y} \cdot \frac{QDQy}{DQDy} \cdot \frac{x}{y} = 1,$$

and similarly, if we should use the groups of equations into which respectively four and five successive operations enter, we would find the syzygies,

$$\frac{D^4y}{QD^3y} \cdot \frac{Q^4y}{DQ^3y} \cdot \frac{D^2Q^2y}{QDQ^2y} \cdot \frac{Q^2D^2y}{DQD^2y} \cdot \frac{DQ^2Dy}{Q^3Dy} \cdot \frac{QD^2Qy}{D^3Qy} \cdot \frac{DQDQy}{Q^2DQy} \cdot \frac{QDQDy}{D^2QDy} \cdot \frac{x}{y} = 1,$$

and

$$\frac{D^5y}{Q^5y} \cdot \frac{D^3Q^2y}{Q^3D^2y} \cdot \frac{D^2Q^2Dy}{Q^2D^2Qy} \cdot \frac{D^2QDQy}{Q^2DQDy} \cdot \frac{DQ^4y}{QD^4y} \cdot \frac{DQ^2D^2y}{QD^2Q^2y} \cdot \frac{DQD^2Qy}{QDQ^2Dy} \cdot \frac{DQDQDy}{QDQDQy} \cdot \frac{Q^2D^3y}{D^2Q^3y}$$
$$\cdot \frac{QDQ^3y}{DQD^3y} \cdot \frac{Q^2DQ^2y}{D^2QD^2y} \cdot \frac{QDQD^2y}{DQDQ^2y} \cdot \frac{Q^4Dy}{D^4Qy} \cdot \frac{QD^3Qy}{DQ^3Dy} \cdot \frac{Q^3DQy}{D^3QDy} \cdot \frac{QD^2QDy}{DQ^2DQy} \cdot \frac{x}{y} = 1,$$

From these special cases a general law can readily be deduced. In fact if $F_1, F_2, \ldots, F_n$, represent the factors in the numerator, and $f_1, f_2, \ldots, f_n$, the factors of the denominator of the syzygy connecting operations of any order, then

$$\frac{\prod_{i=1}^{n} (DF_i \,.\, Qf_i)}{\prod_{j=1} (QF_j \,.\, Df_j)} = 1$$

represents the syzygy connecting the operations of the next higher order.

11.—THE PROCESS $)_3$ IN THE ORDINARY ALGEBRA.

We have shown how a consistent calculus can be built up in which the process of quotientiation occupies the same place that the process of differentiation occupies in the ordinary calculus. This calculus is not only consistent but could be used, though with less ease than the differential calculus, to represent many facts of the universe. For just as it is arbitrary whether we represent aggregates by sums, or products, or powers of certain of their constituents, so likewise is it arbitrary whether we consider growth as accretion or expansion, and change of any kind as caused by additive or multiplicative processes. In the one case, we give to the element which is considered independent in its change, an additive increment dx, in the other case we give it a multiplicative expansion qx. While it may be convenient in practice to measure absolute change by subtracting the initial state from the final, and relative change by division, it is not necessary to do this. We could measure all primary change by dividing the final state by the initial, and relative change by computing the logarithm of the final state with reference to the initial as base. If logarithms not ratios, powers not multiples, were the functions sought in practice, quotientiation would logically take the place of differentiation, and would probably have preceded it in the evolution of mathematical knowledge.

Let us next consider whether a consistent and possible calculus could be built up involving the process $\frac{d_2 y}{d_2 x}$. $\frac{d_2 y}{d_2 x}$ was defined as $\lim\limits_{h = M_2} \{ F(x\,)_2\, h) \frown_2 F(x) \frown_3 h \}$, an expression which involves the symbol $\frown_3$. The ordinary algebra is not usually extended beyond the process $)_2$ and its inverses, that is, beyond involution and its inverses, evolution and the finding of logarithms, so that the consideration of $\frac{d_2 y}{d_2 x}$ necessitates a short preliminary discussion of the process $)_3$ and its inverses.

The equation $\qquad a\,)_2\, a\,)_2 \ldots .$ to b terms $= a\,)_3\, b$,

when applied to the ordinary algebra, and written at length, becomes $\left(\left(\cdot\cdot\left((a)^a\right)^a\cdot\cdot\right)^a\right)^a$ and we shall denote this by $a^{\underline{|b}}$, where b is the number of a's which are involved. If we interpret generally $a^{\underline{|b-1}}$ to be that function

which, when raised to the a^{th} power, gives $a^{\underline{|b}}$, we have immediately $a^{\underline{|1}} = a$, $a^{\underline{|0}} = \sqrt[a]{a}$, $a^{\underline{|-1}} = \sqrt[a]{\sqrt[a]{a}}$, and generally,

$$a^{\underline{|-b}} = \sqrt[a]{\sqrt[a]{\ldots\sqrt[a]{a}}} = \sqrt[a^{b-1}]{a},$$

where the expression in the middle contains $(b+1)$ radicals, that is, $(b+2)$ letters a.

The first inverse $x = a \smile_3 b$ is defined by $x^{\underline{|b}} = a$, and we shall denote the x thus defined by $x = {}^{\underline{b}|}a$. The second inverse $x = a \frown_3 b$ is defined by $b^{\underline{|x}} = a$, and the x satisfying this equation we denote by $x = rg_b a$, which we read x equals range of a to the scale b. We have then for the complete definition equations of the first and second inverses of $a^{\underline{|b}}$ respectively

$$({}^{\underline{b}|}a)^{\underline{|b}} = a \quad \text{and} \quad b^{\underline{|rg_b a}} = a.$$

$a^{\underline{|b}}$ may also be written $a^{a^{b-1}}$, and using this relation, we may easily establish the following rules of operation:

(a). $(x\,.\,y)^{\underline{|n}} = (x^{\underline{|n}})^{y^{n-1}}.\,(y^{\underline{|n}})^{x^{n-1}}$,

(b). $(x^y)^{\underline{|n}} = (x^{\underline{|(n-1)y+1}})^y$,

(c). $(x^{\underline{|y}})^{\underline{|n}} = x^{\underline{|(n-1)x^{y-1}+y}}$,

(d). $(a^{\underline{|x+y}} = (a^{\underline{|x}})^{a^y} = (a^{\underline{|y}})^{a^x}$,

(e). $\log_a a^{\underline{|xy}} = a^{xy-1}$,

(f). $\log_a (a^{\underline{|x+y}}) = a \log_a (a^{\underline{|x}})\,.\log_a (a^{\underline{|y}}) = \dfrac{a^x\,.\,a^y}{a}$,

(g). $\log_a (a^{\underline{|x}}.\,a^{\underline{|y}})^a = a^x + a^y$,

(h). $\log_{a^{\underline{|n}}} (a^{\underline{|m}}) = a^{m-n}$.

The differential and quotiential coefficients may now be readily computed The work is so simple that we shall give only results.

(i) $y = x^{\underline{|a}}$, $\dfrac{dy}{dx} = yx^{a-2}(1 + \overline{a-1}\log x)$, $\dfrac{qy}{qx} = x^{a-1}(1 + \overline{a-1}\log x)$,

(j) $y = a^{\underline{|x}}$, $\dfrac{dy}{dx} = y \log a \log y$, $\dfrac{qy}{qx} = x \log a \log y$.

(k) $y = {}^{\underline{a}|}x$, $\dfrac{dy}{dx} = \dfrac{y\log y}{x\log x}\cdot\dfrac{1}{1+(a-1)\log y}$, $\dfrac{qy}{qx} = \dfrac{\log y}{\log x}\cdot\dfrac{1}{1+(a-1)\log y}$,

(l) $y = {}^{\underline{x}|}a$, $\dfrac{dy}{dx} = \dfrac{y\,\overline{\log y}^{\,2}}{\log\log y - rg_e a}$, $\dfrac{qy}{qx} = \dfrac{x\,\overline{\log y}^{\,2}}{\log\log y - rg_e a}$,

(m) $y = rg_a x$, $\dfrac{dy}{dx} = \dfrac{1}{x \log x \log a}$, $\qquad \dfrac{qy}{qx} = \dfrac{1}{y \log x \log a}$,

(n) $y = rg_x a$, $\dfrac{dy}{dx} = -\dfrac{1 + (y-1)\log x}{x\,\overline{\log x}^{\,2}}$, $\qquad \dfrac{qy}{qx} = -\dfrac{1 + (y-1)\log x}{y\,\overline{\log x}^{\,2}}$,

12.—The Limiting Process Denoted by $\dfrac{ry}{rx}$.

$a^{\underline{|1}} = a$, hence the equation $a\,)_2\,M_2 = a$ is satisfied and by theorem I, Art. 2,

$$\frac{d_2 y}{d_2 x} = \lim_{h = M_2} \{F(x\,)_2\,h) \frown_2 F(x) \frown_3 h\} = M_2 \frown_3 M_2$$

is indeterminate. When $)_2$, $\frown_2$, $\frown_3$ and M_2 are replaced by their algebraic equivalents, we denote this limit by $\dfrac{ry}{rx}$, so that

$$\frac{ry}{rx} = \lim_{h=1} \{rg_h \log_{F(x)} F(x^h)\},$$

$$= 1 + \lim_{h=1} \{\log_h \log_h \log_{F(x)} F(x^h)\}. \qquad [A]$$

$$= 1 + \lim_{h=1} \left\{ \frac{rg_e \log_{F(x)} F(x^h) - rg_e h}{\log h} \right\}. \qquad [B]$$

Either of the last two forms is convenient for the evaluation of the indeterminate. We consider some special cases first.

(a). $y = x^n$. By formula [B],

$$\frac{ry}{rx} = 1 + \lim_{h=1} \left\{ \frac{rg_e \log_{x^n} (x^h)^n - rg_e h}{\log h} \right\} = 1 + \lim_{h=1} \left\{ \frac{rg_e h - rg_e h}{\log h} \right\} = 1.$$

(b). $y = x^{\underline{|n}}$. By formula [A],

$$\frac{ry}{rx} = 1 + \lim_{h=1} \{\log_h \log_h \log_{x^{\underline{|n}}} (x^h)^{\underline{|n}}\}.$$

Put $x = e^z$, then $\log_{x^{\underline{|n}}} (x^h)^{\underline{|n}} = \log (e^{hz})^{\underline{|n}} / \log (e^z)^{\underline{|n}}$.

Now, by formula (b), sect. 11,

$$(e^{hz})^{\underline{|n}} = (e^{\underline{|(n-1)hz+1}})^{hz} \quad \text{and} \quad (e^z)^{\underline{|n}} = (e^{\underline{|(n-1)z+1}})^z,$$

so that $\log (e^{hz})^{\underline{|n}} = hz e^{(n-1)hz}$ and $\log (e^z)^{\underline{|n}} = z e^{(n-1)z}$.

Hence,
$$\frac{ry}{rx} = 1 + \lim_{h=1} \left\{ \log_h \log_h \left(\frac{hz e^{(n-1)hz}}{z e^{(n-1)z}} \right) \right\},$$

but
$$\lim_{h=1}\left\{\log_h \frac{hze^{(n-1)hz}}{ze^{(n-1)z}}\right\} = \frac{q\,(ze^{(n-1)z})}{qx} = \frac{qz}{qz} + \frac{q\,(e^{(n-1)z})}{qz}$$
$$= 1 + (n-1)\,z = 1 + (n-1)\log x,$$

so that finally
$$\frac{ry}{rx} = 1 + \lim_{h=1}\{\log_h (1 + \overline{n-1}\log x)\} = 1 + \infty = \infty,$$

unless $n=1$, the case which has already been considered.

(c). $y = e^x$. By [A]
$$\frac{ry}{rx} = 1 + \lim_{h=1}\{\log_h \log_h \log_{e^x}(e^{x^h})\}$$
$$\log_{e^x} e^{x^h} = \log_{e^x}(e^x)^{x^{h-1}} = x^{h-1},$$

hence
$$\frac{ry}{rx} = 1 + \lim_{h=1}\{\log_h \log_h (x^{h-1})\}.$$

Put $x = e^z$, then $x^{h-1} = e^{zh}/e^z$,

and
$$\lim_{h=1}\{\log_h x^{h-1}\} = \lim_{h=1}\left\{\log_h \frac{e^{zh}}{e^z}\right\} = \frac{q(e^z)}{qx} = z = \log x,$$

so that finally
$$\frac{ry}{rx} = 1 + \lim_{h=1}\{\log_h \log x\} = 1 + \infty = \infty.$$

In each of the foregoing examples except the first, we found $\frac{ry}{rx} = \infty$. We shall now prove that $y = x^n$ is the only possible function of x for which $\frac{ry}{rx}$ is finite.

Let $y = f(x)$. We have
$$\frac{ry}{rx} = 1 + \lim_{h=1}\{\log_h \log_h \log_{f(x)} f(x^h)\},$$

which, on putting $x = e^z$, becomes
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \log_h \frac{\log f(e^{zh})}{\log f(e^z)}\right\}.$$

now
$$\lim_{h=1}\left\{\log_h \frac{\log f(e^{zh})}{\log f(e^z)}\right\} = \frac{q\,(\log f(e^z))}{qz},$$

so that
$$\frac{ry}{rx} = 1 + \lim_{h=1}\left\{\log_h \left[\frac{q\,(\log f(e^z))}{qz}\right]\right\},$$

and since $\log_1 a = \infty$, $a \neq 1$, this expression $= \infty$ unless $\dfrac{q \log f(e^z)}{qz} = 1$. But

$$\frac{q \log f(e^z)}{qz} = \frac{q \log f(e^z)}{qf(e^z)} \cdot \frac{qf(e^z)}{qe^z} \cdot \frac{qe^z}{qz},$$

and $$\frac{q \log f(e^z)}{qf(e^z)} = \frac{1}{\log f(e^z)}, \quad \frac{qf(e^z)}{qe^z} = \frac{qy}{qx}, \quad \frac{qe^z}{qz} = z = \log x,$$

so that $$\frac{q \log f(e^z)}{qz} = \frac{\log x}{\log y} \cdot \frac{qy}{qx} = \frac{x \log x}{y \log y} \cdot \frac{dy}{dx},$$

and that this may equal unity, we must have $\dfrac{dy}{dx} = \dfrac{\log (y^y)}{\log (x^x)}$, which, therefore, is the condition that $\dfrac{ry}{rx} \neq \infty$.

This equation of condition may easily be solved, for, on separating the variables, we have

$$\frac{dy}{\log (y^y)} = \frac{dx}{\log (x^x)},$$

hence $$\int \frac{dy}{\log (y^y)} - \int \frac{dx}{\log (x^x)} = c, \text{ and since} \int \frac{dy}{\log (y^y)} = \log \log y,$$

the equation of condition reduces to

$$\log \log y - \log \log x = c.$$

This equation is transcendental in form only, for

$$\begin{aligned} \log \log y &= c + \log \log x, \\ \log y &= e^{c + \log \log x} = e^c . \log x, \\ y &= e^{e^c . \log x} = (e^{\log x})^{e^c} = x^{e^c} = x^n. \end{aligned}$$

where $n = e^c$. We conclude, therefore, that $\dfrac{ry}{rx} = 1$ or ∞, according as $y =$ or is different from x^n, a fact which must render inadequate any calculus involving only these limiting processes.

13.—DE MORGAN'S EXTENSION OF THE ALGEBRAIC PROCESSES.

It would appear from the foregoing articles that the algebraic processes do not admit of a generalization of the process of differentiation, that in fact the only possible extension of any consequence is quotientiation. This inference is

erroneous. The fault lies with the extension of the algebraic processes themselves. We shall see that $a^{\log b}$ as well as a^b may be looked upon as the $a\,)_2\, b$ of the ordinary algebra, and that when the former is chosen, a consistent calculus may be built up in which $\frac{d_2 y}{d_2 x}$ is the limiting process employed. In fact, by choosing properly the functions

$$\begin{array}{llll} \omega_0(b)\ , & \omega_1(b)\ , & \ldots\ldots & \omega_n(b)\ , \\ \omega_{-1}(b), & \omega_{-2}(b), & \ldots\ldots & \omega_{-n}(b), \end{array}$$

which enter into the definition equations of associated processes (art. 1), we shall arrive at a set of processes in which a $\frac{d_n y}{d_n x}$ exists for any two consecutive processes $)_{n-1}$ and $)_n$.

De Morgan* has shown how to extend the algebraic processes both forward and backward without violating the principle of the permanence of the formal laws of the addition and multiplication processes. *If, in the equations in art.* 1, *we put* $\omega_0(b) = b$, *the* $)_0$ *and* $)_1$ *processes can at once be identified with the ordinary arithmetic addition and multiplication. If, furthermore, we put*

$$\begin{aligned} \omega_1(b) &= b\,))\, r, \\ \omega_2(b) &= b\,\overline{))\, r}^{\,2}, \\ &\vdots \\ \omega_n(b) &= b\,\overline{))\, r}^{\,n}, \end{aligned}$$

where $))$ *represents any process such that*

$$a\,)_1\, b\,))\, r = a\,))\, r\,)_0\,(b\,))\, r),$$

and
$$b\,\overline{))\, r}^{\,n} = b\,))\, r\,))\, r\,))\ \ldots\ldots \text{ to } b+1 \text{ terms},$$

then
$$a\,)_{n+1}\, b\,))\, r = a\,))\, r\,)_n\,(b\,))\, r). \qquad \text{[I]}$$

The proof of this proposition is easy. Let us assume that for some n

$$a\,)_n\, b\,))\, r = a\,))\, r\,)_{n-1}\,(b\,))\, r),$$

then, by definition,

$$a\,)_{n+1}\, b = a\,)_n\, a\,)_n\ \ldots\ldots \text{ to } b\,\overline{))\, r}^{\,n} \text{ terms}.$$

* De Morgan's "Extension of the Algebraic Processes," by Christine Ladd, American Journal of Mathematics (1880), vol. 3.

hence

$$
\begin{aligned}
a\,)_{n+1} b\,))\,r &= (a\,)_n\, a\,)_n \ \dots\ \text{to } \overline{b\,))\,r}^{\,n} - 1 \text{ terms})\,))\,r\,)_{n-1}(a\,))\,r),\\
&= (a\,)_n\, a\,)_n \ \dots\ \text{to } \overline{b\,))\,r}^{\,n} - 2 \text{ terms})\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1}(a\,))\,r),\\
&\ \ \vdots\\
&= a\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1} \ \dots\ \text{to } \overline{b\,))\,r}^{\,n} \text{ terms},\\
&= a\,))\,r\,)_{n-1}(a\,))\,r)\,)_{n-1} \ \dots\ \text{to } \overline{(b\,))\,r)\,))\,r}^{\,n-1} \text{ terms},\\
&= a\,))\,r\,)_n(b\,))\,r) \text{ by definition.}
\end{aligned}
$$

The proposition holds, therefore, for any n if it holds for some particular n, but by hypothesis it holds when $n = 1$.

By a reapplication of [I] we have

$$
\begin{aligned}
a\,)_n \overline{b\,))\,r}^{\,n} &= a\,))\,r\,)_{n-1}\overline{(b\,))\,r)\,))\,r}^{\,n-1},\\
&= \overline{a\,))\,r}^{\,2}\,)_{n-2}\overline{(\overline{b\,))\,r}^{\,2})\,))\,r}^{\,n-2},\\
&\ \ \vdots\\
&= \overline{a\,))\,r}^{\,p}\,)_{n-p}\overline{(\overline{b\,))\,r}^{\,p})\,))\,r}^{\,n-p}, \qquad p \leqq n, \qquad \text{[II]}
\end{aligned}
$$

when $p = n$, this becomes

$$
a\,)_n \overline{b\,))\,r}^{\,n} = \overline{a\,))\,r}^{\,n}\,)_0(\overline{b\,))\,r}^{\,n}). \qquad \text{[III]}
$$

By definition, page 2,

$$
a\,))\,r \,\underline{\smile}\, r = a,
$$

so that we may put $a \,\underline{\smile}\, r = \overline{a\,))\,r}^{\,-1}$ without destroying the law of indices. Similarly,

$$
a \,\underline{\smile}\, r \,\underline{\smile}\, r \,\underline{\smile}\, \underset{-\!-\ -n}{\text{to } n} + 1 \text{ terms} = a \,\overline{\underline{\smile}\, r}^{\,n} = \overline{a\,))\,\boldsymbol{r}}^{\,-n}.
$$

If, now, we take $\omega_{-n}(b) = b\,))\,r$, all of the processes in art. 1 come under the single formula

$$
a\,)_n\, a\,)_n \ \dots\ \text{to } \overline{b\,))\,r}^{\,n} \text{ terms} = a\,)_{n+1} b,
$$

when n may now be negative as well as positive. The fact that n is positive does not enter into the proof of formulæ [I], [II] and [III], so that these also hold when n is negative.

Formula [III] shows how any process, whether positive or negative, may be expressed in terms of the addition process, hence, the existence of a)) process

necessitates the existence of all the) processes. Now, $\log_b a$ satisfies the definition equation of the)) process for

$$\log(m\,.\,n) = \log m + \log n,$$

hence *all the processes, both positive and negative exist and may be written out in terms of the addition process.*

Again, equation III, shows that the process $)_n$ is commutative; it has, therefore, but one inverse $\smile_n$, and since

$$\log(a \div b) = \log a - \log b,$$

the equation

$$a \smile_1 b\,))\,r = a\,))\,r \smile_0 (b\,))\,r)$$

is satisfied, from which equations I, II, III follow with $)_n$ replaced by $\smile_n$. We have finally, in accordance with the definitions of article 1,

$$a \top_n b\,\overline{))\,r}^{\,n} = a\,\overline{))\,r}^{\,n} \top (b\,\overline{))\,r}^{\,n}). \qquad [IV]$$

If the natural logarithmic process is the)) process employed,

$$a\,))\,r = \log a, \quad a\,\overline{))\,r}^{\,-1} = e^a,$$

and [IV] becomes

$$a \top_n b = e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e^{(\log^n a \mp \log^n b)}}}}}}},$$

$$a \top_{-n} b = \log^n\left[e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e^a}}}}}} \mp e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e^b}}}}}}\right],$$

according as the index of $\top$ is positive or negative, and $e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e}}}}}}$ signifies $e^{e^{\cdot^{\cdot^{\cdot}}}}$ to n e's just as $\log^n$ signifies log log .... to n logs. If we write

$$e^a = \log^{-1}a,\ e^{e^a} = \log^{-2}a, \text{ and generally, } e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e^a}}}}}} = \log^{-n}a,$$

$a\top_n b$, and $a\top_{-n} b$ may be written

$$\begin{aligned} a\top_n b &= \log^{-n}[\log^n a \mp \log^n b], \\ a\top_{-n} b &= \log^n[\log^{-n} a \mp \log^{-n} b]. \end{aligned}$$

In particular, we have

$$\begin{aligned} a\top_4 b &= \log^{-4}[\log^4 a \mp \log^4 b], \\ a\top_3 b &= \log^{-3}[\log^3 a \mp \log^3 b], \\ a\top_2 b &= \log^{-2}[\log^2 a \mp \log^2 b] = a^{1 \overset{+}{\times} \log b}, \\ a\top_1 b &= \log^{-1}[\log a \mp \log b] = a \overset{\div}{\times} b, \\ a\top_0 b &= a \mp b, \\ a\top_{-1} b &= \log[\log^{-1} a \mp \log^{-1} b], \\ a\top_{-2} b &= \log^2[\log^{-2} a \mp \log^{-2} b], \\ a\top_{-3} b &= \log^3[\log^{-3} a \mp \log^{-3} b], \\ a\top_{-4} b &= \log^4[\log^{-4} a \mp \log^{-4} b], \\ &\text{etc.} \end{aligned} \qquad [A]$$

Finally, *each of the processes* $\top_n$ *admits a modulus.* To show this, we put in [III], $b = \overline{M_0))\,r}^{-n}$, M_0 being the modulus of the process $\top_0$. We get

$$\begin{aligned} a\top_n\left(\overline{M_0))\,r}^{-n}\right)\overline{))\,r}^{n} &= a\,\overline{))\,r}^{n}\top\left(\overline{M_0))\,r}^{-n}\,\overline{))\,r}^{n}\right), \\ &= a\,\overline{))\,r}^{n}\top M_0 = a\,\overline{))\,r}^{n}, \end{aligned}$$

hence $a\top_n\left(\overline{M_0))\,r}^{-n}\right) = a$, that is, $\overline{M_0))\,r}^{-n} = M_n$.

But M_0 exists, being 0, hence

$$M_n = \log^{-n} 0, \quad M_{-n} = \log^n 0,$$

according as the process in question belongs to the positive or negative series In particular,

$$\begin{aligned} &M_0 = 0, \quad M_1 = 1, \quad M_2 = e, \quad M_3 = e^e, \quad M_4 = e^{e^e}, \quad \text{etc.,} \\ &M_{-1} = -\infty, \quad M_{-2} = \log(-\infty)^* \quad M_{-3} = \log^2(-\infty), \quad \text{etc.} \end{aligned}$$

14.—GENERAL EXTENSION OF THE DIFFERENTIATION PROCESS.

The theorem last proven is of utmost importance. For, since every process $)_n$, as now defined, admits a modulus, it follows from theorem I, art. 2, that

$$\frac{d_n y}{d_n x} = M_{n-1} \frown_n M_{n-1}$$

---

* Here $\log(-\infty) = \log(-1) + \log \infty$, where $\log \infty = \lim\limits_{x=\infty}[\log x]$, x being a positive real number.

is indeterminate, and it remains to show that this expression can be evaluated, i. e. that it has a distinctive determinate form in every case. This problem we will now consider. We begin with some special cases.

(a). $\dfrac{d_1 y}{d_1 x}, \quad y = F(x).$

Referring to the forms [A], art. 13, we have

$$\begin{aligned}
\frac{d_1 y}{d_1 x} &= \lim_{h=M_1} \left\{ F(x\,)_1 h) \frown_1 F(x) \frown_2 h \right\} = \lim_{h=1} \left\{ \log^{-2} \left[\log^2 \frac{F(xh)}{F(x)} - \log^2 h \right] \right\} \\
&= \lim_{h=1} \left\{ \log^{-1} \frac{\log \dfrac{F(xh)}{F(x)}}{\log h} \right\} = \lim_{h=1} \left\{ \log^{-1} \log_h \frac{F(xh)}{F(x)} \right\} \\
&= \log^{-1} \left\{ \lim_{h=1} \left[\log_h \frac{F(xh)}{F(x)} \right] \right\} = \log^{-1} \frac{qy}{qx} \\
&= \log^{-1} \frac{d \log y}{d \log x} = e^{\frac{x}{y} \cdot \frac{dy}{dx}}.
\end{aligned}$$

(b). $\dfrac{d_2 y}{d_2 x}.$

$$\begin{aligned}
\frac{d_2 y}{d_2 x} &= \lim_{h=M_2} \{ F(x\,)_2 h) \frown_2 F(x) \frown_3 h \} \\
&= \lim_{h=e} \left\{ \{ \log^{-3} \{ \log^3 \log^{-2} [\log^2 F(\log^{-2} \overline{\log^2 x + \log^2 h}) - \log^2 F(x)] - \log^3 h) \} \right\}.
\end{aligned}$$

Let us put $\log h = k$, $\log x = z$, then $k = 1$ when $h = e$, and

$$F(\log^{-2} \overline{\log^2 x + \log^2 h}) = F(e^{zk}),$$

$$\log^{-2} [\log^2 F(e^{zk}) - \log^2 (F(x)] = \log^{-1} \frac{\log F(e^{zk})}{\log F(e^z)},$$

$$\log^{-3} \left\{ \log^3 \log^{-1} \frac{\log F(e^{zk})}{\log F(e^z)} - \log^3 h \right\} = \log^{-2} \log_k \frac{\log F(e^{zk})}{\log F(e^z)},$$

hence

$$\begin{aligned}
\frac{d_2 y}{d_2 x} &= \log^{-2} \left\{ \lim_{k=1} \left[\log_k \frac{\log F(e^{zk})}{\log F(e^z)} \right] \right\} = \log^{-2} \frac{q \log F(e^z)}{qz} \\
&= \log^{-2} \frac{q \log y}{q \log x} = \log^{-2} \frac{d \log^2 y}{d \log^2 x} = e^{e^{\left[\frac{\log x}{\log y} \cdot \frac{x}{y} \cdot \frac{dy}{dx}\right]}}.
\end{aligned}$$

(c). $\dfrac{d_{-2}y}{d_{-2}x}$.

$$\frac{d_{-2}y}{d_{-2}x}=\lim_{h=M_{-2}}\{F(x)_{-2}h)\frown_{-2}F(x)\frown_{-1}h\}$$

$$=\lim_{h=\log^2 0}\left\{\log\{\log^{-1}\log^2[\log^{-2}F(\log^2\overline{\log^{-2}x+\log^{-2}h})-\log^{-2}F(x)]-\log^{-1}h\}\right\}.$$

Put $\log^{-3}h=k$, $\log^{-3}x=z$, then $k=1$ as $h=\log^2 0$, also

$$F(\log^2\overline{\log^{-2}x+\log^{-2}h})=F(\log^3 zk),$$

$$\log^2[\log^{-2}F(\log^3 zk)-\log^{-2}F(x)]=\log^3\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\},$$

$$\log\left\{\log^{-1}\log^3\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\}-\log^{-1}h\right\}=\log^2\log_k\left\{\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right\},$$

hence,

$$\frac{d_{-2}y}{d_{-2}x}=\log^2\left\{\lim_{k=1}\left[\log_k\frac{\log^{-3}F(\log^3 zk)}{\log^{-3}F(\log^3 z)}\right]\right\}=\log^2\frac{q\log^{-3}F(\log^3 z)}{qz}$$

$$=\log^2\frac{q\log^{-3}y}{q\log^{-3}x}=\log^2\frac{d\log^{-2}y}{d\log^{-2}x}$$

$$=\log^2\left[\frac{e^{e^y}}{e^{e^x}}\cdot\frac{e^y}{e^x}\cdot\frac{dy}{dx}\right].$$

(d.) *The general case* $\dfrac{d_n y}{d_n x}$, *for any positive or negative index.*

Whether n is positive or negative, we have

$$\frac{d_n y}{d_n x}=\lim_{h=M_n}\{F(x)_n h)\frown_n F(x)\frown_{n+1}h\}$$

$$=\lim_{h=M_n}\Big\{\log^{-(n+1)}\{\log^{n+1}\log^{-n}[\log^n F(\log^{-n}\overline{\log^n x+\log^n h})$$

$$-\log^n F(x)]-\log^{n+1}h\}\Big\}$$

$$=\lim_{h=\log^{-n}0}\left\{\log^{-n}\frac{\log^n\log^{-(n-1)}\dfrac{\log^{n-1}F(\log^{-(n-1)}\overline{\log^{n-1}x.\ \log^{n-1}h})}{\log^{n-1}F(x)}}{\log^n h}\right\}.$$

If we now put $\log^{n-1}h=k$, $\log^{n-1}x=z$, then as h approaches the limit $\log^{-n}0$, k approaches the limit unity; also

$$\frac{\log^{n-1}F(\log^{-(n-1)}\overline{\log^{n-1}x\,.\,\log^{n-1}h})}{\log^{n-1}F(x)}=\frac{\log^{n-1}F(\log^{-(n-1)}\overline{zk})}{\log^{n-1}F(\log^{-(n-1)}z)},$$

so that

$$\frac{d_n y}{d_n x} = \lim_{k=1} \left\{ \log^{-n} \frac{\log \dfrac{\log^{n-1} F(\log^{-(n-1)} \overline{zk})}{\log^{n-1} F(\log^{-(n-1)} z)}}{\log k} \right\}$$

$$= \log^{-n} \left\{ \lim_{k-1} \left[\log_k \frac{\log^{n-1} F(\log^{-(n-1)} \overline{zk})}{\log^{n-1} F(\log^{-(n-1)} z)} \right] \right\}$$

$$= \log^{-n} \frac{q \log^{n-1} F(x)}{qz} = \log^{-n} \frac{q \log^{n-2} y}{q \log^{n-2} x}$$

$$= \log^{-n} \frac{d \log^n y}{d \log^n x}. \qquad \text{[A]}$$

When n is positive, this may be written

$$\frac{d_n y}{d_n x} = e^{\cdot^{\cdot^{(n)^{\cdot^{\cdot^{e\left[\frac{\log^{n-1} x}{\log^{n-1} y} \cdot \frac{\log^{n-2} x}{\log^{n-2} y} \cdots \frac{\log x}{\log y} \cdot \frac{x}{y} \cdot \frac{dy}{dx}\right]}}}}}} \qquad \text{[B]}$$

When n is negative, say $n = - m$,

$$\frac{d_{-m} y}{d_{-m} x} = \log^m \frac{d \log^{-m} y}{d \log^{-m} x} = \log^m \left\{ \frac{e^{\cdot^{\cdot^{(m)^{\cdot^{\cdot^{e^y}}}}}}}{e^{\cdot^{\cdot^{(m)^{\cdot^{\cdot^{e^x}}}}}}} \cdot \frac{e^{\cdot^{\cdot^{(m-1)^{\cdot^{\cdot^{e^y}}}}}}}{e^{\cdot^{\cdot^{(m-1)^{\cdot^{\cdot^{e^x}}}}}}} \cdots \cdot \frac{e^y}{e^x} \cdot \frac{dy}{dx} \right\}. \qquad \text{[C]}$$

We shall call $\frac{d_n y}{d_n x}$ the *ratient* of the n^{th} order, so that the ratient of the zero order is the ordinary differential coefficient, the ratient of the first positive order is the exponential of the quotiential coefficient.

Formulæ [B] and [C] express the ratients of the n^{th} order in terms of differential coefficients, but it is possible likewise to express ratients of the n^{th} order in terms of the ratients of any lower or higher order. We have in fact

$$\frac{d_n y}{d_n x} = \log^{-n}\frac{d\log^n y}{d\log^n x} = \log^{-n}\left[\frac{d\log^n y}{d\log^{n-1} y}\cdot\frac{d\log^{n-1} y}{d\log^{n-1} x}\cdot\frac{d\log^{n-1} x}{d\log^n x}\right]$$

$$= \log^{-n}\left[\frac{\log^{n-1} x}{\log^{n-1} y}\cdot\frac{d\log^{n-1} y}{d\log^{n-1} x}\right] = \log^{-n}\left[\frac{\log^{n-1} x}{\log^{n-1} y}\cdot\log^{n-1}\log^{-(n-1)}\frac{d\log^{n-1} y}{d\log^{n-1} x}\right]$$

$$= \log^{-n}\left[\frac{\log^{n-1} x}{\log^{n-1} y}\cdot\log^{n-1}\frac{d_{n-1} y}{d_{n-1} x}\right],$$

similarly,

$$\frac{d_{n-1} y}{d_{n-1} x} = \log^{-(n-1)}\left[\frac{\log^{n-2} x}{\log^{n-2} y}\cdot\log^{n-2}\frac{d_{n-2} y}{d_{n-2} x}\right],$$

so that

$$\frac{d_n y}{d_n x} = \log^{-n}\left[\frac{\log^{n-1} x}{\log^{n-1} y}\cdot\frac{\log^{n-2} x}{\log^{n-2} y}\cdot\log^{n-2}\frac{d_{n-2} y}{d_{n-2} x}\right],$$

and, generally,

$$\frac{d_n y}{d_n x} = \log^{-n}\left[\frac{\log^{n-1} x}{\log^{n-1} y}\cdot\frac{\log^{n-2} x}{\log^{n-2} y}\cdots\cdot\frac{\log^{n-r} x}{\log^{n-r} y}\cdot\log^{n-r}\frac{d_{n-r} y}{d_{n-r} x}\right]. \qquad [D]$$

Again,

$$\frac{d_n y}{d_n x} = \log^{-n}\frac{d\log^n y}{d\log^n x} = \log^{-n}\left[\frac{d\log^n y}{d\log^{n+1} y}\cdot\frac{d\log^{n+1} y}{d\log^{n+1} x}\cdot\frac{d\log^{n+1} x}{d\log^n x}\right]$$

$$= \log^{-n}\left[\frac{\log^n y}{\log^n x}\cdot\frac{d\log^{n+1} y}{d\log^{n+1} x}\right] = \log^{-n}\left[\frac{\log^n y}{\log^n x}\cdot\log^{n+1}\log^{-(n+1)}\frac{d\log^{n+1} y}{d\log^{n+1} x}\right]$$

$$= \log^{-n}\left[\frac{\log^n y}{\log^n x}\cdot\log^{n+1}\frac{d_{n+1} y}{d_{n+1} x}\right],$$

and, generally,

$$\frac{d_n y}{d_n x} = \log^{-n}\left[\frac{\log^n y}{\log^n x}\cdot\frac{\log^{n+1} y}{\log^{n+1} x}\cdots\cdot\frac{\log^{n+r-1} y}{\log^{n+r-1} x}\cdot\log^{n+r}\frac{d_{n+r} y}{d_{n+r} x}\right]. \qquad [E]$$

Successive ratients may be readily expressed in a compact form. For instance, whether n is positive or negative,

$$\frac{d_n^2 y}{d_n x^2} = \frac{d_n\left[\frac{d_n y}{d_n x}\right]}{d_n x} = \log^{-n}\frac{d\log^n\left[\frac{d_n y}{d_n x}\right]}{d\log^n x} = \log^{-n}\frac{d\left(\frac{d\log^n y}{d\log^n x}\right)}{d\log^n x}$$

$$= \log^{-n}\frac{d^2\log^n y}{(d\log x)^2},$$

and hence, assuming that

$$\frac{d_n^{r-1} y}{d_n x^{r-1}} = \log^{-n} \frac{d^{r-1} \log^n y}{(d \log x)^{r-1}},$$

it follows by induction that generally

$$\frac{d_n^r y}{d_n x^r} = \log^{-n} \frac{d^r \log^n y}{(d \log x)^r}. \qquad [F]$$

15.—GENERAL RATIENTIATION FORMULÆ.

Let us now form the ratient of the m^{th} order of the $u \mathsf{T}_n v$, u and v being functions of x. We have from the preceding paragraph,

for $m \overline{\overline{>}} n$,

$$\frac{d_m (u \mathsf{T}_n v)}{d_m x} = \log^{-m} \frac{d \log^m (u \mathsf{T}_n v)}{d \log^m x}$$

$$= \log^{-m} \left[\frac{d \log^m (u \mathsf{T}_n v)}{d \log^{m-1} (u \mathsf{T}_n v)} \cdot \frac{d \log^{m-1} (u \mathsf{T}_n v)}{d \log^{m-2} (u \mathsf{T}_n v)} \cdots \frac{d \log^{n+1} (u \mathsf{T}_n v)}{d \log^n (u \mathsf{T}_n v)} \cdot \frac{d \log^n (u \mathsf{T}_n v)}{d \log^m x} \right]$$

$$= \log^{-m} \left[\frac{1}{\log^{m-1} (u \mathsf{T}_n v)} \cdot \frac{1}{\log^{m-2} (u \mathsf{T}_n v)} \cdots \frac{1}{\log^n (u \mathsf{T}_n v)} \cdot \frac{d (\log^n u \mp \log^n v)}{d \log^m x} \right],$$

now

$$\frac{d \log^n u}{d \log^n x} = \frac{d \log^n u}{d \log^{n+1} u} \cdot \frac{d \log^{n+1} u}{d \log^{n+2} u} \cdots \frac{d \log^{m-1} u}{d \log^m u} \cdot \frac{d \log^m u}{d \log^m x}$$

$$= \log^n u \,.\, \log^{n+1} u \cdots \log^{m-1} u \,.\, \log^m \frac{d_m u}{d_m x},$$

so that finally

$$\frac{d_m (u \mathsf{T}_n v)}{d_m x} = \log^{-m} \frac{\log^n u \,.\, \log^{n+1} u \cdots \log^{m-1} u \,.\, \log^m \dfrac{d_m u}{d_m x} \mp \log^n v \,.\, \log^{n+1} v \cdots \log^{m-1} v \,.\, \log^m \dfrac{d_m v}{d_m x}}{\log^n (u \mathsf{T}_n v) \log^{n+1} (u \mathsf{T}_n v) \cdots \log^{m-1} (u \mathsf{T}_n v)}; \qquad [A]$$

for $m \leqq n$,

$$\frac{d_m(u \top_n v)}{d_m x} = \log^{-m}\frac{d\log^m(u\top_n v)}{d\log^m x}$$

$$= \log^{-m}\left[\frac{d\log^m(u\top_n v)}{d\log^{m+1}(u\top_n v)}\cdot\frac{d\log^{m+1}(u\top_n v)}{d\log^{m+2}(u\top_n v)}\cdots\cdot\frac{d\log^{n-1}(u\top_n v)}{d\log^{n}(u\top_n v)}\cdot\frac{d\log^{n}(u\top_n v)}{d\log^m x}\right]$$

$$= \log^{-m}\left[\log^m(u\top_n v)\,.\log^{m+1}(u\top_n v)\,\cdots\,\log^{n-1}(u\top_n v)\,.\frac{d(\log^n u \mp \log^n v)}{d\log^m x}\right].$$

and since

$$\frac{d\log^n u}{d\log^m x} = \frac{d\log^n u}{d\log^{n-1} u}\cdot\frac{d\log^{n-1} u}{d\log^{n-2} u}\cdots\cdot\frac{d\log^{m+1} u}{d\log^m u}\cdot\frac{d\log^m u}{d\log^m x}$$

$$= \frac{1}{\log^{n-1} u}\cdot\frac{1}{\log^{n-2} u}\cdots\cdot\frac{1}{\log^m u}\,.\log^m\frac{d_m u}{d_m x},$$

we have

$$\frac{d_m(u\top_n v)}{d_m x} = \log^{-m}\frac{(\log^{n-1}u\,.\log^{n-2}u\,\cdots\,\log^n u)^{-1}\log^m\dfrac{d_m u}{d_m x} \mp (\log^{n-1}v\,.\log^{n-2}v\,\cdots\,\log^m v)^{-1}\log^m\dfrac{d_m v}{d_m x}}{[\log^{n-1}(u\top_n v)\,.\log^{n-2}(u\top_n v)\,\cdots\,\log^m(u\top_n v)]^{-1}}. \qquad [B]$$

Formulæ [A] and [B] enable us to express the ratient of two functions combined by any process, as exponentials or logarithms of sums of multiples of the ratients of the functions taken separately. The extension of the formulæ to the distribution of dm over u, v, w, etc., in $\frac{d_m(u\top_n v\top_n w\top_n,\text{ etc.}}{d_m x}$ is obvious.

When $m = 0$, [A] and [B] become respectively

$$\frac{d(u\top_n v)}{dx} = \frac{\log^n u\,.\log^{n+1}u\,\ldots\,\log^{-1}u\,.\dfrac{du}{dx} \mp \log^n v\,.\log^{n+1}v\,\ldots\,\log^{-1}v\,.\dfrac{dv}{dx}}{\log^n(u\top_n v)\log^{n+1}(u\top_n v)\,\cdots\,\log^{-1}(u\top_n v)},\quad n<0,$$

$$\frac{d(u\top_n v)}{dx} = (u\top_n v)\log(u\top_n v)\,\cdots\,\log^{n-1}(u\top_n v)$$

$$\times\left[\frac{1}{u}\cdot\frac{1}{\log u}\cdots\frac{1}{\log^{n-1}u}\,\frac{du}{dx} \mp \frac{1}{v}\cdot\frac{1}{\log v}\cdots\frac{1}{\log^{n-1}v}\cdot\frac{dv}{dx}\right],\quad n>0,$$

which are the formulæ for the *differential coefficient of two functions combined by the* n^{th} *process.*

When $n=0$, [A] and [B] yield

$$\frac{d_m(u \mp v)}{d_m x}$$

$$= \log^{-m} \frac{u \log u \ldots\ldots \log^{m-1} u \,.\, \log^m \dfrac{d_m u}{d_m x} \mp v \log v \ldots\ldots \log^{m-1} v \,.\, \log^m \dfrac{d_m v}{d_m x}}{(u+v)\log(u+v) \ldots\ldots \log^{m-1}(u+v)} \quad m>0,$$

$$\frac{d_m(u \mp v)}{d_m x} = \log^{-m} \frac{\begin{gathered}(\log^{-1} u \,.\, \log^{-2} u \ldots\ldots \log^m u)^{-1} \log_m \dfrac{d_n u}{d_n x} \\ \mp (\log^{-1} v \,.\, \log^{-2} v \ldots\ldots \log^m v)^{-1} \log^m \dfrac{d_m v}{d_m x}\end{gathered}}{(\log^{-1}(u+v) \log^{-2}(u+v) \ldots\ldots \log^m(u+v))^{-1}}, \quad m<0.$$

formulæ which express *the ratient of a sum of functions in terms of the ratients of the functions taken separately.*

Putting $n=1$, we get corresponding expressions for the *ratient of a product.*

When $n=1$, $m=0$, we get from [B] the *differential coefficient of a product.*

$n=2$, $m=0$ gives, after replacing $\log v$ by v in the final result, and taking the lower sign,

$$\frac{du^v}{dx} = vu^{v-1}\frac{du}{dx} + \log u \,.\, u^v \frac{dv}{dx}.$$

These special cases suffice to indicate the extreme generality of formulæ [A] and [B].

Finally, it is not difficult to show that for $m=n$ [A] or [B] leads to

$$\frac{d_n(u \top_n v)}{d_n x} = \frac{d_n u}{d_n x} \top_n \frac{d_n v}{d_n x}, \qquad [C]$$

and for $n=m+1$, [B] becomes, for the lower sign,

$$\frac{d_m(u |_{m+1} v)}{d_m x} = \frac{d_m u}{d_m x} |_{m+1} v |_m \left(\frac{d_m v}{d_m x} |_{m+1} u\right), \qquad [D]$$

and for the upper sign,

$$\frac{d_m(u -_{m+1} v)}{d_m x} = \frac{d_m u}{d_m x} -_{m+1} v -_m \left(\frac{d_m v}{d_m x} |_{m+1} u -_{m+1} \overline{v |_{m+2} \log^{-(n+1)} 2}\right). \qquad [D']$$

[C], [D] and [D′] are generalizations respectively of

$$\frac{d(u \mp v)}{dx} = \frac{du}{dx} \mp \frac{dv}{dx}, \quad \frac{d(u \,.\, v)}{dx} = \frac{du}{dx} \,.\, v + \frac{dv}{dx} \,.\, u,$$

and
$$\frac{d\,(u \div v)}{dx} = \frac{du}{dx} \div v - \frac{dv}{dx} \,.\, u \div v^2.$$

The formulæ
$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}, \quad \frac{dy}{dx} = 1 \div \frac{dx}{dy},$$

assumes the generalized forms
$$\frac{\partial^1_n u}{\partial_n x\, \partial_n y} = \frac{\partial^2_n u}{\partial_n y\, \partial_n x}, \quad \frac{d_n y}{d_n x} = M_{n+1} -_{n+1} \frac{d_n x}{d_n y},$$

and in fact every theorem or formula in the ordinary calculus has its analogue in the calculus of ratients.

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CONTENTS.